Section 6.1 **Definition of the Laplace Transform.**

Definition 1. Let $f(x)$ be a function on $[0,\infty)$. The **Laplace transform** of f is the function F defined by the integral

$$F(s) = \int\limits_0^\infty f(t)\mathrm{e}^{-st}dt.$$

The domain of $F(s)$ is all the values of s for which integral exists. The Laplace transform of f is denoted by both F and $\mathcal{L}\{f\}$.

Notice, that integral in definition is **improper** integral.

$$\int\limits_0^\infty f(t)\mathrm{e}^{-st}dt = \lim_{N\to\infty}\int\limits_0^N f(t)\mathrm{e}^{-st}dt$$

whenever the limit exists.

Example 1. Determine the Laplace transform of the given function.

1. $f(t) = 1$, $t \geq 0$.

2. $f(t) = t$, $t \geq 0$.

3. $f(t) = \mathrm{e}^{at}$, where a is a constant.

4. $f(t) = \begin{cases} t^2, & 0 < t < 1, \\ 1, & 1 \leq t \leq 2, \\ 1 - t, & 2 < t. \end{cases}$

Brief table of Laplace transform

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
1	$\frac{1}{s}$, $s > 0$
e^{at}	$\frac{1}{s-a}$, $s > a$
t^n, $n = 1, 2, ...$	$\frac{n!}{s^{n+1}}$, $s > 0$
$\sin bt$	$\frac{b}{s^2 + b^2}$, $s > 0$
$\cos bt$	$\frac{s}{s^2 + b^2}$, $s > 0$
$e^{at} t^n$, $n = 1, 2, ...$	$\frac{n!}{(s-a)^{n+1}}$, $s > a$
$e^{at} \sin bt$	$\frac{b}{(s-a)^2 + b^2}$, $s > a$
$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2 + b^2}$, $s > a$

The important property of the Laplace transform is its **linearity**.

Theorem 1. (linearity of the transform) Let f_1 and f_2 be functions whose Laplace transform exist for $s > \alpha$ and c_1 and c_2 be constants. Then, for $s > \alpha$,

$$\mathcal{L}\{c_1 f_1 + c_2 f_2\} = c_1 \mathcal{L}\{f_1\} + c_2 \mathcal{L}\{f_2\}.$$

Example 2. Determine $\mathcal{L}\{10 + 5\mathrm{e}^{2t} + 3\cos 2t\}$.

Existence of the transform.

There are functions for which the improper integral in Definition 1 fails to converge for any value of s. For example, no Laplace transform exists for the function e^{t^2}. Fortunately, the set of the functions for which the Laplace transform is defined includes many of the functions.

Definition 2. A function f is said to be **piecewise continuous on a finite interval** $[a, b]$ if f is continuous at every point in $[a, b]$, except possibly for a finite number of points at which $f(t)$ has a jump discontinuity.

A function $f(x)$ is said to be **piecewise continuous on** $[0, \infty)$ if $f(t)$ is piecewise continuous on $[0, N]$ for all $N > 0$.

Definition 3. A function $f(t)$ is said to be of **exponential order** α if there exist positive constants T and M s.t.

$$|f(t)| \leq M\mathrm{e}^{\alpha t}, \text{ for all } t \geq T.$$

Theorem 2. If $f(t)$ is piecewise continuous on $t \to \infty$ and of exponential order α, then $\mathcal{L}\{f\}(s)$ exists for $s > \alpha$.

Properties of Laplace transform

1. $\mathcal{L}\{f + g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$
2. $\mathcal{L}\{cf\} = c\mathcal{L}\{f\}$ for any constant c
3. $\mathcal{L}\{\mathrm{e}^{at} f\}(s) = F(s - a)$
4. $\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$
5. $\mathcal{L}\{f''\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)$
6. $\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$
7. $\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \dfrac{d^n}{ds^n}(\mathcal{L}\{f(t)\})(s)$

Example 3. Find the Laplace transform of the function.

1. $f(t) = t^2 e^{-2t}$

2. $f(t) = t \sin(5t)$

Inverse Laplace Transform.

Definition 3. Given a function $F(s)$, if there is a function $f(t)$ that is continuous on $[0, \infty)$ and satisfies

$$\mathcal{L}\{f\}(s) = F(s),$$

then we say that $f(t)$ is the **inverse Laplace transform** of $F(s)$ and employ the notation $f(t) = \mathcal{L}^{-1}\{F\}(t)$.
Example 3. Determine the inverse Laplace transform of the given function.

1. $F(s) = \dfrac{2}{s^3}$.

2. $F(s) = \dfrac{2}{s^2 + 4}$.

3. $F(s) = \dfrac{s + 1}{s^2 + 2s + 10}$.

4. $F(s) = \dfrac{s}{s^2 + s - 2}$,

5. $F(s) = \dfrac{3s^2 + 5s + 3}{s^4 - s^2}$