Section 6.1 **Definition of the Laplace Transform.**

Definition 1. Let $f(x)$ be a function on $[0, \infty)$. The **Laplace transform** of f is the function F defined by the integral

$$\boxed{\mathcal{L}\{f\}(s) = F(s) = \int_0^{\infty} f(t)e^{-st}dt.}$$

The domain of $F(s)$ is all the values of s for which integral exists. The Laplace transform of f is denoted by both F and $\mathcal{L}\{f\}$.

Notice, that integral in definition is **improper** integral.

$$\int_0^{\infty} f(t)e^{-st}dt = \lim_{N\to\infty} \int_0^{N} f(t)e^{-st}dt$$

whenever the limit exists.

Example 1. Determine the Laplace transform of the given function.

1. $f(t) = 1$, $t \geq 0$.

$$\mathcal{L}\{1\} = \int_0^\infty 1 \cdot e^{-st}\,dt = \lim_{a\to\infty} \int_0^a e^{-st}\,dt = \lim_{a\to\infty} \left[-\frac{1}{s}\right] e^{-st}\Big|_0^a$$

$$= \frac{-1}{s} \lim_{a\to\infty} \Big[\underbrace{\cancel{e^{-as}}}_{0,\ \text{whenever } s>0} - 1\Big] = \frac{1}{s}, \quad s>0$$

$$\boxed{\mathcal{L}\{1\} = \frac{1}{s},\ s>0}$$

2. $f(t) = t,\ t \geq 0.$

$$\mathcal{L}\{t\} = \int_0^\infty t e^{-st}\,dt = \lim_{a\to\infty}\int_0^a t e^{-st}\,dt \qquad \left|\begin{array}{ll} u = t & v' = e^{-st} \\ u' = 1 & v = -\frac{1}{s}e^{-st} \end{array}\right|$$

$$= \lim_{a\to\infty}\left[-\frac{t}{s}e^{-st}\Big|_0^a - \int_0^a -\frac{1}{s}e^{-st}\,dt\right]$$

$$= \lim_{a\to\infty}\left[-\frac{a}{s}e^{-as} + \frac{1}{s}\int_0^a e^{-st}\,dt\right]$$

$$= \lim_{a\to\infty}\left[-\frac{a}{s}e^{-as} - \frac{1}{s^2}e^{-st}\Big|_0^a\right]$$

$$= \lim_{a\to\infty}\left[-\frac{a}{s}e^{-as} - \frac{1}{s^2}e^{-as} + \frac{1}{s^2}\right]$$

$$= -\overset{0\cdot\infty}{\lim_{a\to\infty}\frac{a}{s}e^{-as}} - \frac{1}{s^2}\overbrace{\lim_{a\to\infty}e^{-as}}^{0,\text{ if } s>0} + \frac{1}{s^2}$$

$$= -\frac{1}{s}\lim_{a\to\infty}\frac{a}{e^{as}} + \frac{1}{s^2} = -\frac{1}{s}\overbrace{\lim_{a\to\infty}\frac{1}{se^{as}}}^{0,\text{ if } s>0} + \frac{1}{s^2} = \frac{1}{s^2}, \quad s>0$$

$$\boxed{\mathcal{L}\{t\} = \frac{1}{s^2},\ s>0}$$

3. $f(t) = e^{at}$, a is a constant

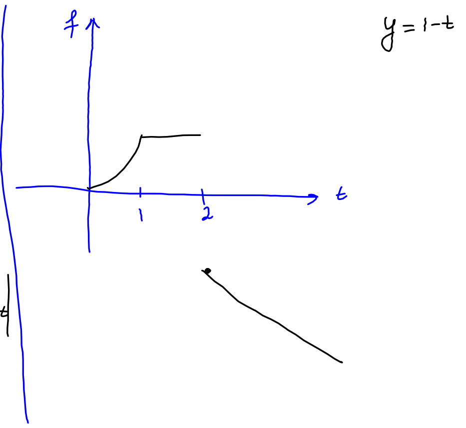
$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{at} \cdot e^{-st}\,dt = \int_0^\infty e^{(a-s)t}\,dt$$

$$= \lim_{N\to\infty} \int_0^N e^{(a-s)t}\,dt = \frac{1}{a-s} \lim_{N\to\infty} \left[\xcancel{e^{(a-s)N}} - 1 \right] \quad 0, \text{ if } a-s<0 \text{ or } s>a$$

$$= \frac{1}{s-a}$$

$$\boxed{\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s>a}$$

4. $f(t) = \begin{cases} t^2, & 0 < t < 1, \\ 1, & 1 \le t \le 2, \\ 1-t, & 2 < t. \end{cases}$



$$\mathcal{L}\{f\} = \int_0^\infty f(t)e^{-st}dt$$

$$= \int_0^1 t^2 e^{-st}dt + \int_1^2 1\cdot e^{-st}dt + \int_2^\infty (1-t)e^{-st}dt$$

$$\left|\begin{matrix} u = t^2 & v' = e^{-st} \\ u' = 2t & v = -\frac{1}{s}e^{-st} \end{matrix}\right| \qquad \left|\begin{matrix} u = 2t & v' = e^{-st} \\ u' = 2 & v = -\frac{1}{s}e^{-st} \end{matrix}\right|$$

$$= -\frac{t^2}{s}e^{-st}\Big|_0^1 + \int_0^1 \frac{2t}{s}e^{-st}dt - \frac{1}{s}e^{-st}\Big|_1^2$$

$$+ \lim_{N\to\infty}\int_2^N (1-t)e^{-st}dt$$

$$\left|\begin{matrix} u = 1-t & v' = e^{-st} \\ u' = -1 & v = -\frac{1}{s}e^{-st} \end{matrix}\right|$$

$$= -\cancel{\frac{e^{-s}}{s}} + \left(-\frac{2t}{s}e^{-st}\Big|_0^1 + \frac{2}{s}\int_0^1 e^{-st}dt\right) - \frac{1}{s}e^{-2s} + \cancel{\frac{1}{s}e^{-s}}$$

$$+ \lim_{N\to\infty}\left[-\frac{1-t}{s}e^{-st}\Big|_2^N - \frac{1}{s}\int_0^N e^{-st}dt\right]$$

$$= -\frac{2}{s}e^{-s} - \frac{2}{s^2}e^{-st}\Big|_0^1 - \frac{1}{s}e^{-2s} + \lim_{N\to\infty}\left[-\cancel{\frac{1-N}{s}e^{-sN}} - \frac{1}{s}e^{-2s}\right]$$

0, if $s>0$

$$+ \frac{1}{s^2}\lim_{N\to\infty} e^{-st}\Big|_0^N$$

$$= -\frac{2}{s}e^{-s} - \frac{2}{s^2}e^{-s} + \frac{2}{s^2} - \frac{1}{s}e^{-2s} - \frac{1}{s}e^{-2s} + \frac{1}{s^2}\left[\cancel{\lim_{N\to\infty} e^{-sN}}\right] - \frac{1}{s^2}$$

0, if $s>0$

$$= \boxed{-\frac{2}{s}e^{-s} - \frac{2}{s^2}e^{-s} - \frac{2}{s}e^{-2s} + \frac{1}{s^2}}, \quad s>0$$

Brief table of Laplace transform

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
1	$\frac{1}{s}$, $s > 0$
e^{at}	$\frac{1}{s-a}$, $s > a$
t^n, $n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}$, $s > 0$
$\sin bt$	$\frac{b}{s^2+b^2}$, $s > 0$
$\cos bt$	$\frac{s}{s^2+b^2}$, $s > 0$
$e^{at}t^n$, $n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}$, $s > a$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$, $s > a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$, $s > a$

The important property of the Laplace transform is its **linearity**.

Theorem 1. (linearity of the transform) Let f_1 and f_2 be functions whose Laplace transform exist for $s > \alpha$ and c_1 and c_2 be constants. Then, for $s > \alpha$,

$$\mathcal{L}\{c_1 f_1 + c_2 f_2\} = c_1 \mathcal{L}\{f_1\} + c_2 \mathcal{L}\{f_2\}.$$

Example 2. Determine $\mathcal{L}\{10 + 5e^{2t} + 3\cos 2t\}. = \mathcal{L}\{10\} + \mathcal{L}\{5e^{2t}\} + \mathcal{L}\{3\cos 2t\}$

$$= 10\,\mathcal{L}\{1\} + 5\,\mathcal{L}\{e^{2t}\} + 3\,\mathcal{L}\{\cos 2t\}$$

$$= \boxed{10 \cdot \frac{1}{s} + 5 \cdot \frac{2}{s-2} + 3\,\frac{s}{s^2+4}}$$

Existence of the transform.

There are functions for which the improper integral in Definition 1 fails to converge for any value of s. For example, no Laplace transform exists for the function e^{t^2}. Fortunately, the set of the functions for which the Laplace transform is defined includes many of the functions.

Definition 2. A function f is said to be **piecewise continuous on a finite interval** $[a, b]$ if f is continuous at every point in $[a, b]$, except possibly for a finite number of points at which $f(t)$ has a jump discontinuity.

A function $f(x)$ is said to be **piecewise continuous on** $[0, \infty)$ if $f(t)$ is piecewise continuous on $[0, N]$ for all $N > 0$.

Definition 3. A function $f(t)$ is said to be of **exponential order** α if there exist positive constants T and M s.t.

$$|f(t)| \leq Me^{\alpha t}, \text{ for all } t \geq T.$$

Theorem 2. If $f(t)$ is piecewise continuous on $t \to \infty$ and of exponential order α, then $\mathcal{L}\{f\}(s)$ exists for $s > \alpha$.

Properties of Laplace transform

1. $\mathcal{L}\{f+g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$
2. $\mathcal{L}\{cf\} = c\mathcal{L}\{f\}$ for any constant c
3. $\mathcal{L}\{e^{at}f\}(s) = F(s-a)$, $F(s) = \mathcal{L}\{f\}$.
4. $\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$
5. $\mathcal{L}\{f''\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)$
6. $\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$
7. $\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n}(\mathcal{L}\{f(t)\})(s)$

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Example 3. Find the Laplace transform of the function.

$\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$

1. $f(t) = t^2 e^{-2t}$

$$\mathcal{L}\{t^2 e^{-2t}\} = \frac{2!}{(s+2)^{2+1}} = \boxed{\frac{2}{(s+2)^3}}$$

2. $f(t) = t\sin(5t)$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[F(s)], \quad F(s) = \mathcal{L}\{f(t)\}$$

$$\mathcal{L}\{\sin 5t\} = \frac{5}{s^2+25}$$

$$\mathcal{L}\{t \sin 5t\} = (-1)\left(\frac{5}{s^2+25}\right)' = (+1)\ 5\cdot(2s)(s^2+25)^{-2}$$

$$= \boxed{\frac{10s}{(s^2+25)^2}}$$

Inverse Laplace Transform.

Definition 3. Given a function $F(s)$, if there is a function $f(t)$ that is continuous on $[0,\infty)$ and satisfies

$$\mathcal{L}\{f\}(s) = F(s),$$

then we say that $f(t)$ is the **inverse Laplace transform** of $F(s)$ and employ the notation $f(t) = \mathcal{L}^{-1}\{F\}(t)$.

Example 3. Determine the inverse Laplace transform of the given function.

1. $F(s) = \dfrac{2}{s^3}$.

$\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$, $\mathcal{L}\{t^2\} = \dfrac{2!}{s^{2+1}} = \dfrac{2}{s^3}$

$$\boxed{\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} = t^2}$$

2. $F(s) = \dfrac{2}{s^2+4}$.

$\mathcal{L}\{\sin bt\} = \dfrac{b}{s^2+b^2}$

$$\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$= \boxed{\sin 2t}$$

3. $F(s) = \dfrac{s+1}{s^2+2s+10}$. $= \dfrac{s+1}{(s^2+2s)+10} = \dfrac{s+1}{(s^2+2s+1)-1+10} = \dfrac{s+1}{(s+1)^2+9} = \dfrac{(s+1)}{(s+1)^2+3^2}$

$$\boxed{\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+3^2}\right\} = e^{-t}\cos 3t}$$

4. $F(s) = \frac{s}{s^2+s-2}$,

$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

Partial fractions

$$\frac{s}{s^2+s-2} = \frac{s}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1}$$

$$s = A(s-1) + B(s+2)$$

$$s=1: \quad 1 = 3B \Rightarrow B = \frac{1}{3}$$

$$s=-2: \quad -2 = -3A \Rightarrow A = \frac{2}{3}$$

$$\frac{s}{s^2+s-2} = \frac{2}{3}\frac{1}{s+2} + \frac{1}{3}\frac{1}{s-1}$$

$$\mathcal{L}^{-1}\left\{\frac{2}{3}\frac{1}{s+2} + \frac{1}{3}\frac{1}{s-1}\right\} = \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$= \boxed{\frac{2}{3}e^{-2t} + \frac{1}{3}e^{t}}$$

5. $F(s) = \frac{3s^2+5s+3}{s^4-s^2} = \frac{3s^2+5s+3}{s^2(s^2-1)} = \frac{3s^2+5s+3}{s^2(s-1)(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s+1}$

$$= \frac{A(s-1)(s+1)s + B(s-1)(s+1) + Cs^2(s+1) + Ds^2(s-1)}{s^2(s-1)(s+1)}$$

$$3s^2+5s+3 = As^3 - As + Bs^2 - B + Cs^3 + Cs^2 + Ds^3 - Ds^2$$

$s^3: \quad 0 = A + D + C$

$s^2: \quad 3 = B + C - D$

$s: \quad 5 = -A \Rightarrow \boxed{A = 5}$

$1: \quad 3 = -B \Rightarrow \boxed{B = -3}$

$D + C = -A = +5$

$+ \; C - D = 3 - B = 6$

$2C = 11 \Rightarrow \boxed{C = \frac{11}{2}}$

$D = 5 - C = 5 - \frac{11}{2} = \boxed{-\frac{1}{2} = D}$

$$\mathcal{L}^{-1}\left\{ -\frac{5}{s} - \frac{3}{s^2} + \frac{11}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1} \right\}$$

$$= \boxed{-5 - 3t + \frac{11}{2}e^{t} - \frac{1}{2}e^{-t}}$$