Section 6.2 **Solution of initial value problems.**

To solve an initial value problem:

- Take the Laplace transform of both sides of the equation.
- Use the properties of the Laplace transform and the initial conditions to obtain an equation for the Laplace transform of the solution and then solve this equation for the transform.
- Determine the inverse Laplace transform of the solution.

Important formulas:

$$\mathcal{L}\{y'\}(s) = s\mathcal{L}\{y\}(s) - y(0)$$
$$\mathcal{L}\{y''\}(s) = s^2\mathcal{L}\{y\}(s) - sy(0) - y'(0)$$

Example 1. Solve the initial value problem.

1. $y'' + 6y' + 9y = 0$, $y(0) = -1$, $y'(0) = 6$

Set $\mathcal{L}\{y(t)\} = Y(s)$

$\mathcal{L}\{y'(t)\} = sY(s) - \cancel{y(0)}^{-1} = sY(s) + 1$

$\mathcal{L}\{y''(t)\} = s^2Y(s) - s\cancel{y(0)}^{-1} - \cancel{y'(0)}^{6} = s^2Y(s) + s - 6$

$\mathcal{L}\{y'' + 6y' + 9y = 0\}$

$\mathcal{L}\{y''\} + 6\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = 0$

$s^2Y(s) + s - 6 + 6(sY(s) + 1) + 9Y(s) = 0,$

Solve for $Y(s)$:

$(s^2 + 6s + 9)Y(s) + s - 6 + 6 = 0$

$Y(s)(s^2 + 6s + 9) = -s$

$y(t) = \mathcal{L}^{-1}\{Y(s)\} = -\mathcal{L}^{-1}\left\{\frac{s}{s^2+6s+9}\right\}$

$$\frac{s}{s^2+6s+9} = \frac{s}{(s+3)^2} = \frac{A}{s+3} + \frac{B}{(s+3)^2} = \frac{A(s+3)+B}{(s+3)^2}$$

$s = As + (3A + B)$

$s: \quad A = 1$

$1: \quad 3A + B = 0 \Rightarrow B = -3A = -3$

$$y(t) = -\mathcal{L}^{-1}\left\{\frac{1}{s+3} - \frac{3}{(s+3)^2}\right\} = \boxed{-e^{-3t} + 3te^{-3t}}$$

2. $y'' + 6y' + 5y = 12e^t$, $y(0) = -1$, $y'(0) = 7$.

$$\mathcal{L}\{y(t)\} = Y(s)$$

$$\mathcal{L}\{y'(t)\} = sY(s) - \overset{-1}{\cancel{y(0)}} = sY(s)+1$$

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - s\overset{-1}{\cancel{y(0)}} - \overset{7}{\cancel{y'(0)}} = s^2Y(s)+s-7$$

$$\mathcal{L}\{y''+6y'+5y\} = \mathcal{L}\{12e^t\} = \frac{12}{s-1}$$

$$\mathcal{L}\{y''\} + 6\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = \frac{12}{s-1}$$

$$\underbrace{s^2Y(s)+s-7}_{\mathcal{L}\{y''\}} + 6\underbrace{(sY(s)+1)}_{\mathcal{L}\{y'\}} + 5\underbrace{Y(s)}_{\mathcal{L}\{y\}} = \frac{12}{s-1}$$

$$Y(s)\underbrace{(s^2+6s+5)}_{\text{L.H.S. of the auxiliary equation}} + s-7+6 = \frac{12}{s-1}$$

$$Y(s)(s^2+6s+5) = \frac{12}{s-1} - s+1 = \frac{12}{s-1} - (s-1) = \frac{12-(s-1)^2}{s-1} = \frac{12-s^2+2s-1}{s-1}$$

$$Y(s) = \frac{11-s^2+2s}{(s-1)(s^2+6s+5)} = \frac{11-s^2+2s}{(s-1)(s+5)(s+1)}$$

Partial fractions: $\dfrac{11-s^2+2s}{(s-1)(s+5)(s+1)} = \dfrac{A}{s-1} + \dfrac{B}{s+5} + \dfrac{C}{s+1}$

$$11-s^2+2s = A(s+5)(s+1) + B(s^2-1) + C(s-1)(s+5)$$

$s=1$: $11-1+2 = A(6)(2) \Rightarrow 12 = 12A \Rightarrow \boxed{A=1}$

$\boxed{\mathcal{L}\{e^{at}\} = \frac{1}{s-a}}$

$s=-1$: $11-1-2 = C(-2)(4) \Rightarrow 8 = -8C \Rightarrow \boxed{C=-1}$

$s=-5$: $11-25-10 = B(25-1) \Rightarrow -24 = 24B \Rightarrow \boxed{B=-1}$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-1} - \frac{1}{s+5} - \frac{1}{s+1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\} - \mathcal{L}\left\{\frac{1}{s+1}\right\}$$

$$= e^t - e^{-5t} - e^{-t}$$

3. $y'' - 2y' + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$.

$$\mathcal{L}\{y(t)\} = Y(s)$$

$$\mathcal{L}\{y'(t)\} = sY(s) - \cancel{y(0)}^{\,0} = sY(s)$$

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - s\cancel{y(0)}^{\,0} - \cancel{y'(0)}^{\,1} = s^2Y(s) - 1$$

$$\mathcal{L}\{y'' - 2y' + 2y\} = \mathcal{L}\{e^{-t}\}$$

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{e^{-t}\}$$

$$s^2Y(s) - 1 - 2sY(s) + 2Y(s) = \frac{1}{s+1}$$

$$Y(s)(s^2 - 2s + 2) = \frac{1}{s+1} + 1 = \frac{s+2}{s+1}$$

$$Y(s) = \frac{s+2}{(s+1)(s^2-2s+2)} = \frac{A}{s+1} + \frac{Bs+C}{s^2-2s+2}$$

$b^2 - 4ac = 4 - 8 = -4 < 0$

$$s+2 = A(s^2-2s+2) + (s+1)(Bs+C)$$

$$s+2 = As^2 - 2As + 2A + Bs^2 + Bs + Cs + C$$

$$\begin{cases} s^2: & 0 = A + B \Rightarrow B = -A \\ s: & 1 = -2A + B + C \\ 1: & 2 = 2A + C \end{cases}$$

$$\begin{cases} 1 = -3A + C \Rightarrow C = 1 + 3A \\ 2 = 2A + C \end{cases}$$

$$2 = 2A + 1 + 3A \Rightarrow 5A = 1 \Rightarrow \boxed{A = \frac{1}{5}}$$

$$C = 1 + \frac{3}{5} = \boxed{\frac{8}{5} = C}$$

$$\boxed{B = -\frac{1}{5}}$$

$$Y(s) = \frac{1}{5}\frac{1}{s+1} + \frac{-\frac{1}{5}s + \frac{8}{5}}{s^2-2s+2}$$

$$= \frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{s-8}{\underbrace{(s^2-2s+2)}_{(s^2-2s+1)+1}} = \frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{s-8}{(s-1)^2+1}$$

$$= \frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{(s-1)-7}{(s-1)^2+1}$$

$$= \frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\left[\frac{s-1}{(s-1)^2+1} - \frac{7}{(s-1)^2+1}\right]$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}$$

$$\mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$$

$$Y(s) = \frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{s-1}{(s-1)^2+1} + \frac{7}{5}\frac{1}{(s-1)^2+1}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+1}\right\} + \frac{7}{5}\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+1}\right\}$$

$$= \boxed{\frac{1}{5}e^{-t} - \frac{1}{5}e^{t}\cos t + \frac{7}{5}e^{t}\sin t}$$