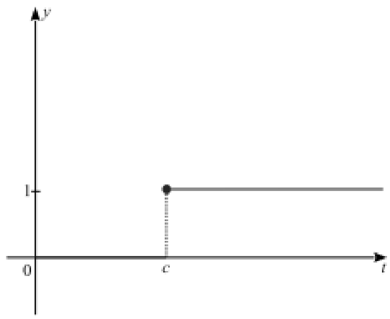
Section 6.3 **Step functions.**

Definition. The **unit step function** or **Heaviside function** is defined by

$$u_c(t) = \begin{cases} 0, & t < c, \\ 1, & t \geq c, \end{cases} \qquad c \geq 0.$$



Graph of $y = u_c(t)$

Example 1. Express $f(t)$ in terms of $u_c(t)$ if

1. $f(t) = \begin{cases} 0, & 0 \leq t < 3, \\ -2, & 3 \leq t < 5, \\ 2, & 5 \leq t < 7, \\ 1, & t \geq 7. \end{cases}$

$$f(t) = 0 + u_3(t)(-2-0) + u_5(t)(2-(-2)) + u_7(t)(1-2)$$
$$= \boxed{-2u_3(t) + 4u_5(t) - u_7(t)}$$

2. $f(t) = \begin{cases} t, & 0 \leq t < 1, \\ t-1, & 1 \leq t < 2, \\ t-2, & 2 \leq t < 3, \\ 0, & t \geq 3. \end{cases} = t + u_1(t)\left[(t-1) - t\right] + u_2(t)\left[(t-2) - (t-1)\right] + u_3(t)\left[0 - (t-2)\right]$

$$= \boxed{t - u_1(t) - u_2(t) - (t-2)u_3(t)}$$

$$\boxed{\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}, \qquad s > 0.}$$

Theorem. If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$, and c is a positive constant, then

$$\boxed{\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s),} \qquad s > a.$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$\boxed{u_c(t)f(t-c) = \mathcal{L}^{-1}\{e^{-cs}F(s)\}}$$

Example 2. Find the Laplace transform of the following functions:

1. $f(t) = u_1(t) + 2u_3(t) - 6u_4(t)$

$$\mathcal{L}\{f(t)\} = \boxed{\frac{e^{-s}}{s} + 2\frac{e^{-3s}}{s} - 6\frac{e^{-4s}}{s}}$$

2. $g(t) = (t-3)u_2(t) - (t-2)u_3(t)$

$$= u_2(t)\left[(t-2)-1\right] - u_3(t)\left[(t-3)+1\right]$$

$$= (t-2)u_2(t) - u_2(t) - (t-3)u_3(t) - u_3(t)$$

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{(t-2)u_2(t)\} - \mathcal{L}\{u_2(t)\} - \mathcal{L}\{(t-3)u_3(t)\} - \mathcal{L}\{u_3(t)\}$$

$$f(t-2) = t-2,\quad f(t-3) = t-3 \Rightarrow f(t) = t,\quad \mathcal{L}\{t\} = \frac{1}{s^2}$$

$$= \boxed{e^{-2s}\frac{1}{s^2} - \frac{e^{-2s}}{s} - e^{-3s}\frac{1}{s^2} - \frac{e^{-3s}}{s}}$$

3. $h(t) = \begin{cases} 0, & t < 1, \\ t^2 - 2t + 2, & t \geq 1, \end{cases}$

$$h(t) = 0 + u_1(t)(t^2-2t+2) = u_1(t)(t^2-2t+2) = u_1(t)\left[(t^2-2t+1)+1\right]$$

$$= u_1(t)\left[(t-1)^2+1\right] = u_1(t)(t-1)^2 + u_1(t)$$

$$\mathcal{L}\{h(t)\} = \mathcal{L}\{u_1(t)(t-1)^2\} + \mathcal{L}\{u_1(t)\} = e^{-s}\cdot\mathcal{L}\{t^2\} + \frac{e^{-s}}{s}$$

$$= \boxed{e^{-s}\cdot\frac{2}{s^3} + \frac{e^{-s}}{s}}$$

Example 3. Find the inverse Laplace transform of the given function.

1. $F(s) = \dfrac{e^{-2s}}{s^2+s-2} = e^{-2s} \cdot \dfrac{1}{s^2+s-2}$

$$\mathcal{L}^{-1}\left\{ \frac{1}{s^2+s-2} \right\}$$

$$\frac{1}{s^2+s-2} = \frac{1}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1} = \frac{1}{3}\left(-\frac{1}{s+2} + \frac{1}{s-1} \right)$$

$$1 = A(s-1) + B(s+2)$$

$s=1: \quad 1 = 3B \Rightarrow B = \frac{1}{3}$

$s=-2: \quad 1 = -3A \Rightarrow A = -\frac{1}{3}$

$$\mathcal{L}^{-1}\left\{ \frac{1}{3}\left(\frac{1}{s-1} - \frac{1}{s+2} \right) \right\} = \frac{1}{3}\left[e^{+t} - e^{-2t} \right] = f(t)$$

$$\mathcal{L}^{-1}\left\{ \frac{e^{-2s}}{s^2+s-2} \right\} = u_2(t) f(t-2) = \boxed{u_2(t) \cdot \frac{1}{3}\left(e^{t-2} - e^{-2(t-2)} \right)}$$

2. $F(s) = \dfrac{2(s-1)e^{-2s}}{s^2-2s+2} = e^{-2s} \dfrac{2(s-1)}{s^2-2s+2}$

$$\mathcal{L}^{-1}\left\{ \frac{2(s-1)}{s^2-2s+2} \right\} = \mathcal{L}^{-1}\left\{ \frac{2(s-1)}{s^2-2s+1+1} \right\} = 2\,\mathcal{L}^{-1}\left\{ \frac{s-1}{(s-1)^2+1} \right\} = 2e^{t}\cos t$$

$$\mathcal{L}^{-1}\left\{ e^{-2s} \frac{2(s-1)}{s^2-2s+2} \right\} = u_2(t)(2)e^{t-2}\cos(t-2)$$