Section 6.4 **Differential equations with discontinuous forcing functions.**

Important formulas:

1. $\mathcal{L}\{y'\}(s) = s\mathcal{L}\{y\}(s) - y(0)$
2. $\mathcal{L}\{y''\}(s) = s^2\mathcal{L}\{y\}(s) - sy(0) - y'(0)$
3. $\mathcal{L}\{u_c(t)\} = \dfrac{e^{-cs}}{s}$
4. $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\}$
5. $\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)f(t-c)$, where $F(s) = \mathcal{L}\{f(t)\}$

Example 1. Solve the initial value problem.

$$y'' + 5y' + 6y = g(t), \qquad y(0) = 0, y'(0) = 2,$$

where $g(t) = \begin{cases} 0, & 0 \le t < 1, \\ t, & 1 < t < 5, \\ 1, & 5 < t. \end{cases} = 0 + t\,u_1(t) + (1-t)\,u_5(t) = t\,u_1(t) + (1-t)\,u_5(t)$

$$\mathcal{L}\{y'' + 5y' + 6y\} = \mathcal{L}\{t\,u_1(t) + (1-t)\,u_5(t)\}$$

$$\mathcal{L}\{y(t)\} = Y(s)$$

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY(s)$$

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 2$$

$\boxed{\mathcal{L}\{t\} = \frac{1}{s^2}}$

$$\mathcal{L}\{t\,u_1(t) + (1-t)u_5(t)\} = \mathcal{L}\{[(t-1)+1]u_1(t) - [(t-5)+5-1]u_5(t)\}$$

$$= \mathcal{L}\{(t-1)u_1(t)\} + \mathcal{L}\{u_1(t)\} - \mathcal{L}\{(t-5)u_5(t)\} - 4\mathcal{L}\{u_5(t)\}$$

$$= \frac{e^{-s}}{s^2} + \frac{e^{-s}}{s} - \frac{e^{-5s}}{s^2} - \frac{4e^{-5s}}{s}$$

$$s^2Y(s) - 2 + 5sY(s) + 6Y(s) = e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right) - e^{-5s}\left(\frac{1}{s^2} + \frac{4}{s}\right)$$

$$Y(s)(s^2+5s+6) = e^{-s}\frac{s+1}{s^2} - e^{-5s}\frac{1+4s}{s^2} + 2$$

$$Y(s) = e^{-s}\frac{s+1}{s^2(s^2+5s+6)} - e^{-5s}\frac{1+4s}{s^2(s^2+5s+6)} + \frac{2}{s^2+5s+6}$$

$$Y(s) = e^{-s}\frac{s+1}{s^2(s^2+5s+6)} - e^{-5s}\frac{1+4s}{s^2(s^2+5s+6)} + \frac{2}{s^2+5s+6}$$

$\mathcal{L}^{-1}\{Y(s)\}$:

$$\frac{2}{s^2+5s+6} = \frac{A}{s+2} + \frac{B}{s+3} = \frac{A(s+3)+B(s+2)}{(s+2)(s+3)}$$

$$2 = A(s+3)+B(s+2)$$

$s=-3$: $2 = -B \Rightarrow \boxed{B=-2}$

$s=-2$: $\boxed{2=A}$

$$\mathcal{L}^{-1}\left\{\frac{2}{s^2+5s+6}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s+2} - \frac{2}{s+3}\right\} = 2e^{-2t} - 2e^{-3t}$$

$$\frac{1+4s}{s^2(s^2+5s+6)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} + \frac{D}{s+3}$$

$$1+4s = As(s+2)(s+3) + B(s+2)(s+3) + Cs^2(s+3) + Ds^2(s+2)$$

$s=0$: $1 = 6B \Rightarrow \boxed{B=\frac{1}{6}}$

$s=-2$: $1-8 = C(-2)^2(-2+3) \Rightarrow -7 = 4C \Rightarrow \boxed{C=-\frac{7}{4}}$

$s=-3$: $1-12 = D(-3)^2(-3+2) \Rightarrow -11 = -9D \Rightarrow \boxed{D=\frac{11}{9}}$

$s=-1$: $1-4 = A(-1)(-1+2)(-1+3) + B(-1+2)(-1+3) + C(-1+3) + D(-1+2)$

$$-3 = -2A + 2B + 2C + D$$

$$2A = 3+2B+2C+D$$

$$A = \frac{1}{2}(3+2B+2C+D) = \frac{1}{2}\left(3+\frac{1}{3}-\frac{7}{2}+\frac{22}{9}\right)$$

$$= \frac{1}{2}\,\frac{54+6-63+44}{18} = +\frac{41}{36}$$

$$\mathcal{L}^{-1}\left\{\frac{41}{36}\,\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{6}\,\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{7}{4}\,\frac{1}{s+2}\right\} + \mathcal{L}^{-1}\left\{\frac{11}{9}\cdot\frac{1}{s+3}\right\}$$

$$= \frac{41}{36} + \frac{1}{6}t - \frac{7}{4}e^{-2t} + \frac{11}{9}e^{-3t}$$

$$\mathcal{L}^{-1}\left\{e^{-5s}\frac{1+4s}{s^2(s^2+5s+6)}\right\} = \frac{41}{36} + \frac{1}{6}(t-5) - \frac{7}{4}e^{-2(t-5)} + \frac{11}{9}e^{-3(t-5)}$$

$$\mathcal{L}^{-1}\left\{e^{-s}\frac{s+1}{s^2(s^2+5s+6)}\right\}$$

$$\frac{s+1}{s^2(s^2+5s+6)} = \frac{A}{s}+\frac{B}{s^2}+\frac{C}{s+2}+\frac{D}{s+3}$$

$$s+1 = As(s+2)(s+3)+B(s+2)(s+3)+Cs^2(s+3)+Ds^2(s+2)$$

$s=0$: $1 = 6B \Rightarrow \boxed{B=\frac{1}{6}}$

$s=-2$: $-1 = 4C \Rightarrow \boxed{C=-\frac{1}{4}}$

$s=-3$: $-2 = -9D \Rightarrow \boxed{D=\frac{2}{9}}$

$s=-1$: $0 = -2A+2B+2C+D$

$$2A = 2B+2C+D$$

$$A = B+C+\frac{D}{2} = \frac{1}{6}-\frac{1}{4}+\frac{1}{9} = \frac{6-9+4}{36} = \boxed{\frac{1}{36}=A}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{36}\frac{1}{s}+\frac{1}{6}\frac{1}{s^2}-\frac{1}{4}\frac{1}{s+2}+\frac{2}{9}\frac{1}{s+3}\right\}$$

$$= \frac{1}{36}+\frac{1}{6}t-\frac{1}{4}e^{-2t}+\frac{2}{9}e^{-3t}$$

$$\mathcal{L}^{-1}\left\{e^{-s}\frac{s+1}{s^2(s^2+5s+6)}\right\} = \frac{1}{36}+\frac{1}{6}(t-1)-\frac{1}{4}e^{-2(t-1)}+\frac{2}{9}e^{-3(t-1)}$$

$$\boxed{\begin{aligned} y(t) = \mathcal{L}^{-1}\{Y(s)\} &= \frac{1}{36}+\frac{1}{6}(t-1)-\frac{1}{4}e^{-2(t-1)}+\frac{2}{9}e^{-3(t-1)} \\ &+2e^{-2t}-2e^{-3t}+\frac{41}{36}+\frac{1}{6}(t-5)-\frac{7}{4}e^{-2(t-5)}+\frac{11}{9}e^{-3(t-5)}\end{aligned}}$$