Section 6.5 **Impulse Functions.**

Definition. A **unit impulse function** (**Dirac delta function**) is a function defined by

$$\cancel{\delta(t) = \begin{cases} 1, & t = 0, \\ 0, & t \neq 0. \end{cases}} \qquad \delta(t - t_0) = 0, \quad t \neq t_0$$

Property of the unit impulse function:

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1.$$

$$\boxed{\mathcal{L}\{\delta(t - t_0)\} = e^{-st_0}}$$

Example 1. Solve the initial value problem.

$$y'' + 2y' + 2y = \delta(t - \pi), \qquad y(0) = 1, y'(0) = 0.$$

$$\mathcal{L}\{y(t)\} = Y(s)$$
$$\mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY(s) - 1$$
$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s$$
$$\mathcal{L}\{\delta(t - \pi)\} = e^{-\pi s}$$

$$\mathcal{L}\{y'' + 2y' + 2y\} = \mathcal{L}\{\delta(t - \pi)\}$$
$$s^2Y(s) - s + 2[sY(s) - 1] + 2Y(s) = e^{-\pi s}$$
$$Y(s)(s^2 + 2s + 2) = e^{-\pi s} + s + 2$$

$$\boxed{s^2 + 2s + 2 = [s^2 + 2s + 1] + 1 = (s+1)^2 + 1}$$

$$Y(s) = \frac{e^{-\pi s}}{s^2 + 2s + 2} + \frac{s + 2}{s^2 + 2s + 2}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{e^{-\pi s}\frac{1}{s^2 + 2s + 2}\right\} + \mathcal{L}^{-1}\left\{\frac{s + 2}{s^2 + 2s + 2}\right\}$$

$$= \mathcal{L}^{-1}\left\{e^{-\pi s}\frac{1}{(s+1)^2 + 1}\right\} + \underbrace{\mathcal{L}^{-1}\left\{\frac{(s+1) + 1}{(s+1)^2 + 1}\right\}}_{\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2 + 1}\right\} = e^{-t}\sin t$$

$$\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2 + 1}\right\} = e^{-t}\cos t$$

$$= u_\pi(t)\, e^{-(t-\pi)}\sin(t - \pi) + e^{-t}\sin t + e^{-t}\cos t$$

$$= \boxed{-u_\pi(t)\, e^{-(t-\pi)}\sin t + e^{-t}\sin t + e^{-t}\cos t}$$