Section 6.6 **The convolution integral.**

Theorem. If $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$ both exist for $s > a \geq 0$, then

$$H(s) = F(s)G(s) = \mathcal{L}\{h(t)\}, \qquad s > a,$$

where

$$h(t) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau.$$

The function h is known as a **convolution** of f and g ($h(t) = (f * g)(t)$); the integrals in the formula for $h(t)$ are known as **convolution integrals**.

Properties of the convolution.

1. $f * g = g * f$
2. $f * (g_1 + g_2) = f * g_1 + f * g_2$
3. $(f * g) * h = f * (g * h)$
4. $f * 0 = 0 * f = 0$

Example 1. Find the Laplace transform of the function

$$h\cancel{f}(t) = \int_0^t (t-\tau)^2 \cos 2\tau \, d\tau = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$$

$$f(t-\tau) = (t-\tau)^2 \Rightarrow f(t) = t^2$$
$$g(\tau) = \cos 2\tau \Rightarrow g(t) = \cos 2t$$

$$\mathcal{L}\{h\} = \mathcal{L}\{f\} \cdot \mathcal{L}\{g\} = \mathcal{L}\{t^2\}\, \mathcal{L}\{\cos 2t\}$$

$$= \frac{2}{s^3} \cdot \frac{s}{s^2+4}$$

Example 2. Find $\mathcal{L}^{-1}\left\{\dfrac{1}{s^4(s^2+1)}\right\} = H(s)\cdot G(s)$

$$H(s) = \frac{1}{s^4},\quad G(s) = \frac{1}{s^2+1}$$

$$\boxed{\mathcal{L}\{t^3\} = \frac{3!}{s^4} = \frac{6}{s^4}}$$

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{6}{s^4}\cdot\frac{1}{6}\right\} = \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{6}{s^4}\right\} = \frac{1}{6}t^3$$

$$g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^4(s^2+1)}\right\} = (h*g)(t) = \boxed{\frac{1}{6}\int_0^t (t-\tau)^3 \sin\tau\, d\tau = \frac{1}{6}\int_0^t \tau^3 \sin[t-\tau]\, d\tau}$$

Example 3. Express the solution of the initial value problem

$$\mathcal{L}\{y'' + \omega^2 y\} = \mathcal{L}\{g(t)\} \qquad y(0) = 0, y'(0) = 1$$

in terms of a convolution integral.

$\mathcal{L}\{y(t)\} = Y(s), \quad \mathcal{L}\{g(t)\} = G(s)$

$\mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY(s)$

$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 1$

$s^2Y(s) - 1 + \omega^2 Y(s) = G(s)$

$Y(s)(s^2+\omega^2) = G(s) + 1$

$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{G(s)}{s^2+\omega^2} + \frac{1}{s^2+\omega^2}\right\}$

$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \boxed{\frac{1}{\omega}\int_0^t \sin(\omega\tau)\, g(t-\tau)\, d\tau + \frac{1}{\omega}\sin \omega t}$

$\mathcal{L}^{-1}\left\{\frac{G(s)}{s^2+\omega^2}\right\} = g * \frac{1}{\omega}\sin\omega t$

$= \mathcal{L}^{-1}\left\{G(s) \cdot \frac{1}{s^2+\omega^2}\right\}$

$\mathcal{L}^{-1}\{G(s)\} = g(t)$

$\mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega^2}\right\} = \frac{1}{\omega}\sin\omega t$

$\boxed{\mathcal{L}\{\sin\omega t\} = \frac{\omega}{s^2+\omega^2}}$