Chapter 7. **Systems of first order linear equations.**
Section 7.1 **Introduction**

1. **First-order system** of differential equations:

$$\begin{aligned} x_1' &= F_1(t, x_1, x_2, \ldots, x_n) \\ x_2' &= F_2(t, x_1, x_2, \ldots, x_n) \\ &\vdots \\ x_n' &= F_n(t, x_1, x_2, \ldots, x_n) \end{aligned} \tag{1}$$

2. A set of differentiable functions $x_1(t), x_2(t), \ldots, x_n(t)$ satisfying the system (1) is called a **solution** of the system (1).

3. System of ODE using a **vector notation**:

$$\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} F_1(t, x_1, x_2, \ldots, x_n) \\ F_2(t, x_1, x_2, \ldots, x_n) \\ \vdots \\ F_n(t, x_1, x_2, \ldots, x_n) \end{pmatrix}$$

Then the system (1) can be written as

$$\mathbf{X}' = \mathbf{F}(t, \mathbf{X}). \tag{2}$$

More generally, any differential equation of order n,

$$y^{(n)} = f(t, y, y', y'', \ldots, y^{(n-1)})$$

can be transformed to a system of n differential equations of the first order by introducing derivatives up to order $n-1$ as new variables.

4. To transform the following n-th order IVP,

$$y^{(n)} = f(t, y, y', y'', \dots, y^{(n-1)}),$$

$$(t_0) = \alpha_0, \quad y'(t_0) = \alpha_1, \dots, \quad y^{(n-1)}(t_0) = \alpha_{n-1}$$

into the system we set

$$\begin{aligned} x_1(t) &= y(t) \\ x_2(t) &= y'(t) \\ &\vdots \\ x_n(t) &= y^{(n-1)}(t) \end{aligned}$$

to get

$$\begin{aligned} &x_1' = x_2 \\ &x_2' = x_3 \\ &\vdots \\ &x_n' = f(t, x_1, x_2, \dots, x_n) \end{aligned}$$

subject to

$$x_1(t_0) = \alpha_0, \quad x_2(t_0) = \alpha_1, \dots, \quad x_n(t_0) = \alpha_{n-1}.$$

5. Note, if f depends on t then the system is called **non-autonomous** and the phase portrait (space) in this case is in $\mathbb{R}^{n+1}$. Otherwise (i.e. if f doesn't depend on t)the system is **autonomous** and the phase portrait (space) in this case is in $\mathbb{R}^n$.

6. **Important**: Not any system of n first order ODE comes from a scalar n-th order.

Example 1. Transform the differential equation into a system of first order equations.

(a) $y'' + 5y' - 2y = \sin t$

$x_1(t) = y(t)$

$x_2(t) = y'(t)$ new "variables"

$x_1'(t) = y'(t) = x_2(t) \Rightarrow$

$y'' + 5y' - 2y = \sin t$

$x_2'(t) = (y')' = y'' = \sin t - 5y' + 2y$

$= \sin t - 5x_2(t) + 2x_1(t) \Rightarrow$

$$\begin{cases} x_1'(t) = x_2(t) \\ x_2'(t) = \sin t - 5x_2(t) + 2x_1(t) \end{cases}$$

(b) $y''' + 3y' + y = 4$

$x_1(t) = y(t)$

$x_2(t) = y'(t)$

$x_3(t) = y''(t)$

$y''' = 4 - 3y' + y$

$x_3'(t) = 4 - 3x_2(t) + x_1(t) \Rightarrow$

$$\begin{cases} x_1'(t) = x_2(t) \\ x_2'(t) = x_3(t) \\ x_3'(t) = 4 - 3x_2(t) + x_1(t) \end{cases}$$

Example 2. Transform the initial value problem

$$y'' + .25y' + 4y = 2\cos 3t, \qquad y(0) = 1, y'(0) = -2$$

into a system of 2 first order differential equations.

$x_1(t) = y(t)$
$x_2(t) = y'(t)$ $\Rightarrow$

$y'' = 2\cos 3t - 0.25y' - 4y$
$x_2' = 2\cos 3t - 0.25x_2(t) - 4x_1(t)$ $\Rightarrow$

$$\begin{cases} x_1'(t) = x_2(t) \\ x_2' = 2\cos 3t - 0.25x_2(t) - 4x_1(t) \end{cases}$$

$y(0) = 1 \Rightarrow \boxed{x_1(0) = 1}$
$y'(0) = -2 \Rightarrow \boxed{x_2(0) = -2}$ initial conditions.

7. **Existence and Uniqueness Theorem** for IVP defined by a system: Consider the IVP:

$$\begin{array}{rcl} x_1' & = & F_1(t, x_1, x_2, \ldots, x_n) \\ x_2' & = & F_2(t, x_1, x_2, \ldots, x_n) \\ & \vdots & \\ x_n' & = & F_n(t, x_1, x_2, \ldots, x_n) \\ x_1(t_0) & = & x_1^0 \\ x_2(t_0) & = & x_2^0 \\ & \vdots & \\ x_n(t_0) & = & x_n^0 \end{array} \tag{3}$$

initial conditions.

If each of the functions $F_1, F_2, \ldots, F_n$ and the partial derivatives $\dfrac{\partial F_1}{\partial x_k}, \dfrac{\partial F_2}{\partial x_k}, \ldots, \dfrac{\partial F_n}{\partial x_k}$ $(1 \leq k \leq n)$ are continuous in a region

$$R = \{\alpha < t < \beta, \alpha_1 < x_1 < \beta_1, \alpha_2 < x_1 < \beta_2, \ldots, \alpha_n < x_n < \beta_n\}$$

and the point $(t_0, x_1^0, \ldots, x_n^0)$ belongs to R, then there is an interval $(t_0 - h, t_0 + h)$ in which there exists a unique solution of the IVP (3).

Linear Systems

8. When each of the functions $F_1, F_2, \ldots, F_n$ in (3) is linear in the dependent variables $x_1, \ldots, x_n$, we get a system of linear equations:

$$\begin{cases} x_1' = p_{11}(t)x_1 + p_{12}(t)x_2 + \ldots + p_{1n}(t)x_n + g_1(t) \\ x_2' = p_{21}(t)x_1 + p_{22}(t)x_2 + \ldots + p_{2n}(t)x_n + g_2(t) \\ \quad\vdots \\ x_n' = p_{n1}(t)x_1 + p_{n2}(t)x_2 + \ldots + p_{nn}(t)x_n + g_n(t) \end{cases} \tag{4}$$

When $g_k(t) \equiv 0$ $(1 \le k \le n)$, the linear system (4) is said to be **homogeneous**; otherwise it is **nonhomogeneous.**

9. **Existence and Uniqueness Theorem** for linear IVP:

   If the functions $p_{11}, p_{12}, \ldots, p_{nn}$ and $g_1, \ldots, g_n$ are continuous on an open interval $I = \{t : \alpha < t < \beta\}$, then there exists a unique solution of the system (4) that also satisfies the initial conditions $x_1(t_0) = x_1^0, x_2(t_0) = x_2^0, \ldots, x_n(t_0) = x_n^0$, where t_0 is any point of I. Moreover, the solution exists throughout the interval I.

Matrix Form of A Linear System

10. If X, $P(t)$, and $G(t)$ denote the respective matrices

$$X = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad P(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) & \ldots & p_{1n}(t) \\ p_{21}(t) & p_{22}(t) & \ldots & p_{2n}(t) \\ \vdots & & & \vdots \\ p_{n1}(t) & p_{n2}(t) & \ldots & p_{nn}(t) \end{pmatrix}, \quad G(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{pmatrix},$$

then the system of linear first-order DE (4) can be written as

$$X' = PX + G.$$

If the system is homogeneous, its matrix form is then

$$X' = PX.$$

11. **Example 3.** Express the given system in matrix form:

(a) $\begin{cases} x_1' = x_2 = 0\cdot x_1 + 1\cdot x_2 \\ x_2' = -x_1 = -1\cdot x_1 + 0\cdot x_2 \end{cases}$

$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$, $\vec{x}'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix}$

coefficient matrix $P = \begin{array}{cc} x_1 & x_2 \\ \end{array} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\boxed{\vec{x}'(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{x}} \Rightarrow \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

(b) $\begin{cases} x_1' = x_2 - x_1 + t \\ x_2' = -x_1 + 7x_2 - x_3 - e^t \\ x_3' = 2x_2 - x_3 + \sin t \end{cases}$

$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$, $P = \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \begin{pmatrix} -1 & 1 & 0 \\ -1 & 7 & -1 \\ 0 & 2 & -1 \end{pmatrix}$, $\vec{G}(t) = \begin{pmatrix} t \\ -e^t \\ \sin t \end{pmatrix}$

$$\boxed{\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 7 & -1 \\ 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} t \\ -e^t \\ \sin t \end{pmatrix}}$$