Section 7.3 **Systems of Linear Algebraic Equations. Eigenvalues and Eigenvectors**

Gauss Elimination Method

The Gauss Method is a suitable technique for solving systems of linear equations of any size. A sequence of operations (see below) of the Gauss-Jordan elimination method allows us to obtain at each step an equivalent system - that is, a system having the same solution as the original system.

The operations of the Gauss-Jordan elimination method are

1. Interchange any two equations.
2. Replace an equation by a nonzero multiple of itself.
3. Replace an equation by itself plus a nonzero multiple of any other equation.

An **augmented matrix** that is formed by combining the coefficient matrix and the constant matrix. For example, for the system of linear equations $\begin{cases} 3x_1 & + & 12x_2 & = & 20 \\ 2x_2 & = & x_1 & +7 \end{cases}$ the augmented matrix is $\left(\begin{array}{cc|c} 3 & 12 & \overset{2}{1}0 \\ -1 & +2 & 7 \end{array}\right)$.

The goal of the Gauss Elimination Method is to get the augmented matrix into **Reduced Echelon Form**. A matrix is in row echelon form if

- All nonzero rows (rows with at least one nonzero element) are above any rows of all zeroes (all zero rows, if any, belong at the bottom of the matrix).
- The leading coefficient (the first nonzero number from the left, also called the pivot) of a nonzero row is always strictly to the right of the leading coefficient of the row above it.
- All entries in a column below a leading entry are zeroes (implied by the first two criteria).

To put a matrix in Reduced Form, there are three valid Row Operations:

1. Interchange any two rows ($R_i \leftrightarrow R_j$).
2. Replace any row by a nonzero constant multiple of itself ($R_i \leftrightarrow cR_i$).
3. Replace any row by the sum of that row and a constant multiple of any other row $R_i \leftrightarrow (R_i + cR_j)$.

Eigenvalues and Eigenvectors

Definition. A number λ is called an **eigenvalue** of a matrix A if there exists a **nonzero** vector $\vec{v}$ such that

$$A\vec{v} = \lambda\vec{v},$$

and $\vec{v}$ is called an **eigenvector** corresponding to the eigenvalue λ. **Example.** If A is diagonal matrix,

$$A = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

then the numbers $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues and the vectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \quad v_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},$$

are the corresponding eigenvectors.

Eigenvalue are solutions of the following **characteristic equation (polynomial):**

$$\boxed{\det(A - \lambda I) = 0.}$$

The characteristic equation in the case $n = 2$ can be found as

$$\boxed{\lambda^2 - \text{trace}(A)\lambda + \det(A) = 0.}$$

Remark. For $n \times n$ matrix the characteristic equation is a polynomial equation of degree n. The eigenvectors corresponding to λ can be found by solving the corresponding system of linear equations $(A - \lambda I)\vec{v} = \vec{0}$ (as we will see in the next.

Example 1. Find eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} -2 & 1 \\ 2 & -3 \end{pmatrix}$.

$tr(A) = -2-3 = -5$
$\det(A) = 6-2 = 4$

characteristic polynomial:

$$\det(A-\lambda I) = \begin{vmatrix} -2-\lambda & 1 \\ 2 & -3-\lambda \end{vmatrix} = +(2+\lambda)(3+\lambda) - 2$$

$$= 6+5\lambda+\lambda^2-2 = \underbrace{\lambda^2 + \underbrace{5}_{-tr(A)}\lambda + \underbrace{4}_{\det A} = 0}$$

$$(\lambda+1)(\lambda+4) = 0$$

$\lambda_1 = -1$, $\lambda_2 = -4$ – eigenvalues

Corresponding eigenvectors:

$\lambda_1 = -1$, $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $\vec{v}$ is a solution of the system

$$(A - (-1)I)\vec{v} = \vec{0}$$

$$\begin{pmatrix} -2+1 & 1 \\ 2 & -3+1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -v_1 + v_2 = 0 \\ 2v_1 - 2v_2 = 0 \end{cases} \Rightarrow v_1 = v_2$$

$$\vec{v} = \begin{pmatrix} v_1 \\ v_1 \end{pmatrix} = v_1 \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

$\boxed{\vec{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ is an eigenvector corresponding to } \lambda_1 = -1}$

$\lambda_2 = -4$, $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$, $\vec{w}$ is a solution of

$$(A - (-4)I)\vec{w} = \vec{0}$$

$$\begin{pmatrix} -2+4 & 1 \\ 2 & -3+4 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{cases} 2w_1 + w_2 = 0 \\ 2w_1 + w_2 = 0 \end{cases} \Rightarrow w_2 = -2w_1$$

$$\vec{w} = \begin{pmatrix} w_1 \\ 2w_1 \end{pmatrix} = w_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$\boxed{\vec{x}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \text{ is an eigenvector corresponding to } \lambda_2 = -4}$

Example 2. Given

$$A = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$$

1. Find eigenvalues of A.

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} = -(1-\lambda)(2-\lambda)(1+\lambda) + 2 + 12 - 8(2-\lambda) - 3(1+\lambda) + (1-\lambda)$$

$$= -(1-\lambda^2)(2-\lambda) + 14 - 16 + 8\lambda - 3 - 3\lambda + 1 - \lambda$$

$$= -(2 - 2\lambda^2 - \lambda + \lambda^3) - 4 + 4\lambda$$

$$= -2 + 2\lambda^2 + \lambda - \lambda^3 - 4 + 4\lambda$$

$$= \underline{-\lambda^3 + 2\lambda^2 + 5\lambda - 6 = 0}$$

$$\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$$

$\lambda_1 = 1$ $\quad 1-2-5+6=0$

$$\begin{array}{r} \lambda^2 - \lambda - 6 \\ \lambda - 1 \,\big)\, \overline{\lambda^3 - 2\lambda^2 - 5\lambda + 6} \\ \underline{\lambda^3 - \lambda^2} \\ -\lambda^2 - 5\lambda \\ \underline{-\lambda^2 + \lambda} \\ -6\lambda + 6 \\ \underline{-6\lambda + 6} \\ 0 \end{array}$$

$$\lambda^3 - 2\lambda^2 - 5\lambda + 6 = (\lambda-1)(\lambda^2 - \lambda - 6)$$
$$= (\lambda-1)(\lambda-3)(\lambda+2) = 0$$

$$\boxed{\lambda_1 = 1,\ \lambda_2 = 3,\ \lambda_3 = -2}$$

2. Find eigenvectors of A

$\underline{\lambda_2 = 3}$ $\qquad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$\vec{v}$ is a solution

$$(A - 3I)\vec{v} = \vec{0}$$

$$\begin{pmatrix} 1-3 & -1 & 4 \\ 3 & 2-3 & -1 \\ 2 & 1 & -1-3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -2 & -1 & 4 \\ 3 & -1 & -1 \\ 2 & 1 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

component form: $\begin{cases} -2v_1 - v_2 + 4v_3 = 0 \\ 3v_1 - v_2 - v_3 = 0 \\ 2v_1 + v_2 - 4v_3 = 0 \end{cases}$ $\qquad$ 1st eqn. = (3rd eqn)(-1)

$$\begin{cases} +2v_1 + v_2 - 4v_3 = 0 \Rightarrow & v_2 = 4v_3 - 2v_1 \\ 3v_1 - v_2 - v_3 = 0 \end{cases}$$

$$3v_1 - (4v_3 - 2v_1) - v_3 = 0$$

$$v_1 - 5v_3 = 0 \Rightarrow \quad v_1 = 5v_3$$

$$v_2 = 4v_3 - 2v_1 = 4v_3 - 10v_3 = -6v_3$$

$$\vec{v} = \begin{pmatrix} 5v_3 \\ -6v_3 \\ v_3 \end{pmatrix} = v_3 \begin{pmatrix} 5 \\ -6 \\ 1 \end{pmatrix}$$

$$\boxed{\vec{x}_1 = \begin{pmatrix} 5 \\ -6 \\ 1 \end{pmatrix} \text{ is an eigenvector corresponding to } \lambda = 3}$$