Section 7.4 **Basic theory of systems of first order linear equations.**

Consider a system of n first order linear equations

$$\begin{cases} x_1' &= p_{11}(t)x_1 + \cdots + p_{1n}(t)x_n + g_1(t) \\ &\vdots \\ x_n' &= p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n + g_n(t) \end{cases} \tag{1}$$

System (1) can be written in matrix notation as

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t), \tag{2}$$

where $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \cdots \\ x_n(t) \end{pmatrix}$, $\mathbf{g}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \\ \cdots \\ g_n(t) \end{pmatrix}$, $\mathbf{P}(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) & \cdots & p_{1n}(t) \\ p_{21}(t) & p_{22}(t) & \cdots & p_{2n}(t) \\ \vdots & \vdots & & \vdots \\ p_{n1}(t) & p_{n2}(t) & \cdots & p_{nn}(t) \end{pmatrix}$.

We assume that $\mathbf{P}(t)$ and $\mathbf{g}(t)$ are continuous on some interval $I = [\alpha, \beta]$.

The corresponding system of **homogeneous** equations is

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \tag{3}$$

Once the homogeneous system has been solved, there are several methods that can be used to solve the nonhomogeneous system (2), we'll discuss them later. We use the notation

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \\ \cdots \\ x_{n1}(t) \end{pmatrix}, \cdots, \mathbf{x}^{(k)}(t) = \begin{pmatrix} x_{1k}(t) \\ x_{2k}(t) \\ \cdots \\ x_{nk}(t) \end{pmatrix}, \cdots \tag{4}$$

to designate specific solutions of the system (3).

Theorem 1.(Principle of superposition.) If the vector functions $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are solutions of the system (3), then the linear combination $c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t)$ is also a solution for any constants c_1 and c_2.

Let $\mathbf{x}^{(1)}(t)$, $\mathbf{x}^{(2)}(t)$,..., $\mathbf{x}^{(n)}(t)$ be n solutions of the system (3), and consider the matrix $\mathbf{X}(t)$ whose columns are the vectors $\mathbf{x}^{(1)}(t)$, $\mathbf{x}^{(2)}(t)$,..., $\mathbf{x}^{(n)}(t)$

$$\mathbf{X}(t) = \begin{pmatrix} x_{11}(t) & x_{12}(t) & \cdots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \cdots & x_{2n}(t) \\ \vdots & \vdots & & \vdots \\ x_{n1}(t) & x_{n2}(t) & \cdots & x_{nn}(t) \end{pmatrix}. \tag{5}$$

Definition. The **Wronskian** of the n solutions $\mathbf{x}^{(1)}(t)$, $\mathbf{x}^{(2)}(t)$,..., $\mathbf{x}^{(n)}(t)$ is

$$W\left[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}\right](t) = \det \mathbf{X}(t) \tag{6}$$

The solutions $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ are then **linearly independent** at a point t_0 if and only if $W\left[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}\right](t_0) \neq 0$.

Theorem 2. If the vector functions $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ are linearly independent solutions of the system (3) for each point in the interval I, then each solution $\mathbf{x} = \boldsymbol{\varphi}(t)$ of the system (3) can be expressed as a linear combination of $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$

$$\boldsymbol{\varphi}(t) = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t) \tag{7}$$

in exactly one way.

This means that the equation (7) includes all solutions of the system (3), and it is the **general solution**.

Definition. Any set of solutions $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ of the system (3) that is linearly independent at each point $\alpha < t_0 < \beta$ said to be a **fundamental set of solutions** for that interval.

Theorem 3. If $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ are solutions of the system (3) on the interval I, then in this interval $W\left[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}\right]$ either is identically zero or else never vanishes.

Example 1. Given the system

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \mathbf{x}.$$

Verify that $\mathbf{x}^{(1)}(t) = (e^{2t}, e^{2t}, e^{2t})^T$, $\mathbf{x}^{(2)}(t) = (-e^{-t}, 0, e^{-t})^T$ and $\mathbf{x}^{(3)}(t) = (0, e^{-t}, -e^{-t})^T$ form a fundamental set of solutions.

$$W[\vec{x}^{(1)}, \vec{x}^{(2)}, \vec{x}^{(3)}] = \begin{vmatrix} e^{2t} & -e^{-t} & 0 \\ e^{2t} & 0 & e^{-t} \\ e^{2t} & e^{-t} & -e^{-t} \end{vmatrix}$$

$$= 0 + e^{2t}(-e^{-t})(e^{-t}) + 0 + 0 - e^{2t}(e^{-t})^2 - (e^{-t})^2 e^{2t}$$

$$= -e^{2t}e^{-2t} - e^{2t}e^{-2t} - e^{-2t}e^{2t} = -3 \neq 0$$

$\vec{x}^{(1)}, \vec{x}^{(2)}, \vec{x}^{(3)}$ are linearly independent.

$\vec{x}^{(1)} = \begin{pmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{pmatrix}$ plug into the system.

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{pmatrix} = \begin{pmatrix} 0 + 1\cdot e^{2t} + 1e^{2t} \\ 1\cdot e^{2t} + 0 + 1e^{2t} \\ 1e^{2t} + 1e^{2t} + 0 \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \\ 2e^{2t} \end{pmatrix} = [\vec{x}^{(1)}]'$$

$\vec{x}^{(1)}$ is a solution indeed.

Check if $\vec{x}^{(2)}$ and $\vec{x}^{(3)}$ are also solutions of the system.

Theorem 4. If $\mathbf{x} = \mathbf{u}(t) + i\mathbf{v}(t)$ is a complex-valued solution of the system (3), then its real part $\mathbf{u}(t)$ and its imaginary part $\mathbf{v}(t)$ are also solutions of the system.