Section 7.5 **Homogeneous linear systems with constant coefficients**

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \tag{1}$$

here $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$.

The system (1) is autonomous. Solutions $\mathbf{x}$ for which $\mathbf{A}\mathbf{x} = \mathbf{0}$ correspond to **equilibrium solutions**, and are called **critical points**. We assume that $\det \mathbf{A} \neq 0$ ($\mathbf{A}$ is nonsingular), thus $\mathbf{x} = \mathbf{0} = (0, \dots, 0)$ is the only critical point of the system (1).

If $n = 2$, a solution the system (1) $\mathbf{x} = \boldsymbol{\varphi}(t)$ can be viewed as a parametric representation for a curve in the x_1x_2-plane. This curve can be regarded as a trajectory traversed by a moving particle whose velocity $d\mathbf{x}/dt$ is specified by a differential equation. The x_1x_2-plane is call the **phase plane**, and a representative set of trajectories is called a **phase portrait**.

> If $\mathbf{A}$ has n **distinct real** eigenvalues $\lambda_1, \dots, \lambda_n$ and $\mathbf{v}_1, \dots, \mathbf{v}_n$ are the corresponding eigenvectors, then
>
> $$\{e^{\lambda_1 t}\mathbf{v}_1, \dots, e^{\lambda_n t}\mathbf{v}_n\}$$
>
> is the **fundamental solution set** of the system (1), and the **general solution** of the system (1) is
>
> $$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + \dots + c_n e^{\lambda_n t}\mathbf{v}_n.$$

Example 1. Find the general solution of the system

1. $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x}$

2. $\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \mathbf{x}$

3. $\mathbf{x}' = \begin{pmatrix} -1 & 4 \\ -2 & 5 \end{pmatrix} \mathbf{x}$

If $\mathbf{A}$ has two distinct real eigenvalues λ_1, and λ_2, then

1. If $\lambda_1 > \lambda_2 > 0$, then the point $(0, 0)$ is a **nodal source**, and it is **asymptotically unstable**.
2. If $\lambda_1 < \lambda_2 < 0$, then the point $(0, 0)$ is a **nodal sink**, and it is **asymptotically stable**.
3. If $\lambda_1 < 0 < \lambda_2$, then the point $(0, 0)$ is a **saddle point** , and it is **unstable**.