Section 7.5 **Homogeneous linear systems with constant coefficients**

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \tag{1}$$

here $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$.

The system (1) is autonomous. Solutions $\mathbf{x}$ for which $\mathbf{A}\mathbf{x} = \mathbf{0}$ correspond to **equilibrium solutions**, and are called **critical points**. We assume that $\det \mathbf{A} \neq 0$ ($\mathbf{A}$ is nonsingular), thus $\mathbf{x} = \mathbf{0} = (0, \dots, 0)$ is the only critical point of the system (1).

If $n = 2$, a solution the system (1) $\mathbf{x} = \boldsymbol{\varphi}(t)$ can be viewed as a parametric representation for a curve in the x_1x_2-plane. This curve can be regarded as a trajectory traversed by a moving particle whose velocity $d\mathbf{x}/dt$ is specified by a differential equation. The x_1x_2-plane is call the **phase plane**, and a representative set of trajectories is called a **phase portrait**.

If $\mathbf{A}$ has n **distinct real** eigenvalues $\lambda_1, \dots, \lambda_n$ and $\mathbf{v}_1, \dots, \mathbf{v}_n$ are the corresponding eigenvectors, then

$$\{e^{\lambda_1 t}\mathbf{v}_1, \dots, e^{\lambda_n t}\mathbf{v}_n\}$$

is the **fundamental solution set** of the system (1), and the **general solution** of the system (1) is

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + \dots + c_n e^{\lambda_n t}\mathbf{v}_n.$$

Example 1. Find the general solution of the system

1. $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x}$, $\quad A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$, $\quad A - \lambda I = \begin{pmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{pmatrix}$

Characteristic polynomial of A: $\quad \operatorname{tr} A = 3-2=1$

$\det A = -6+4=-2$

$$\lambda^2 - (\operatorname{tr} A)\lambda + (\det A) = 0$$
$$\lambda^2 - \lambda - 2 = 0$$
$$(\lambda-2)(\lambda+1) = 0$$
$$\lambda_1 = 2, \quad \lambda_2 = -1 \quad \text{- eigenvalues}$$

eigenvectors:

$\lambda_1 = 2$, $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $\vec{v}$ is a solution of the system

$$(A-2I)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(A-2I)\vec{v} = \quad \begin{pmatrix} 3-2 & -2 \\ 2 & -2-2 \end{pmatrix}\vec{v} = \underbrace{\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}}$$

component form: $\begin{cases} v_1 - 2v_2 = 0 \\ 2v_1 - 4v_2 = 0 \end{cases} \qquad v_1 = 2v_2$

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2v_2 \\ v_2 \end{pmatrix} = v_2 \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{\vec{x}_1}$$

$\vec{x}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda = 2$

$\lambda_2 = -1$, $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ is a solution of the system

$(A-(-1)I)\,\vec{w} = \vec{0}$ or $(A+I)\,\vec{w} = \vec{0}$ or $\begin{pmatrix} 3+1 & -2 \\ 2 & -2+1 \end{pmatrix}\vec{w} = \vec{0}$

$$\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

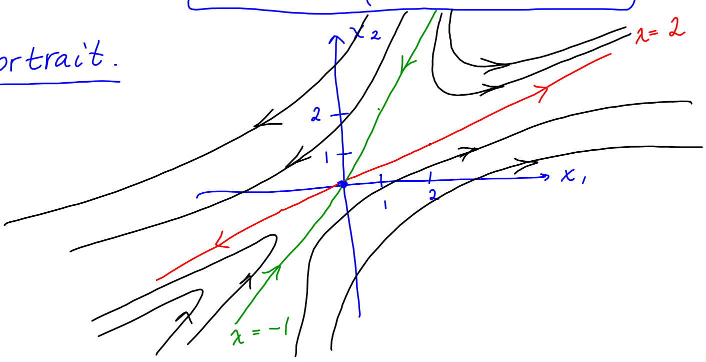
component form: $\begin{cases} 4w_1 - 2w_2 = 0 \\ 2w_1 - w_2 = 0 \end{cases} \qquad w_2 = 2w_1$

$$\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ 2w_1 \end{pmatrix} = w_1 \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{\vec{x}_2}$$

$\vec{x}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector corresponding to $\lambda = -1$.

General solution: $\boxed{\vec{x}(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}}$

Phase portrait.



$\lambda_1 > 0, \ \lambda_2 < 0$

$(0,0)$ is a saddle point

it is unstable

2. $\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}\mathbf{x}$

$A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}$, $\quad tr\ A = 1-4 = -3$

$\qquad\qquad\qquad\qquad det\ A = -4+6 = 2$

Characteristic polynomial: $\lambda^2 + 3\lambda + 2 = 0$

$(\lambda+2)(\lambda+1) = 0$

$\lambda_1 = -2,\ \lambda_2 = -1$ eigenvalues.

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corresponding eigen vectors:

$\lambda_1 = -2$, $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is a solution of the system

$(A - (-2)I)\vec{v} = \vec{0}$ or $(A + 2I)\vec{v} = \vec{0}$ or $\begin{pmatrix} 1+2 & -2 \\ 3 & -4+2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 3 & -2 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

component form $\quad 3v_1 - 2v_2 = 0$ or $3v_1 = 2v_2 \Rightarrow v_2 = \frac{3}{2}v_1$

$\vec{v} = \begin{pmatrix} v_1 \\ \frac{3}{2}v_1 \end{pmatrix} = v_1\begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix} \underline{\underline{v_1 = 2}}$ $\boxed{\vec{x}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \text{ corresponds to } \lambda = -2}$

$\lambda_2 = -1$, $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ is a solution of $(A - (-1)I)\vec{w} = \vec{0}$

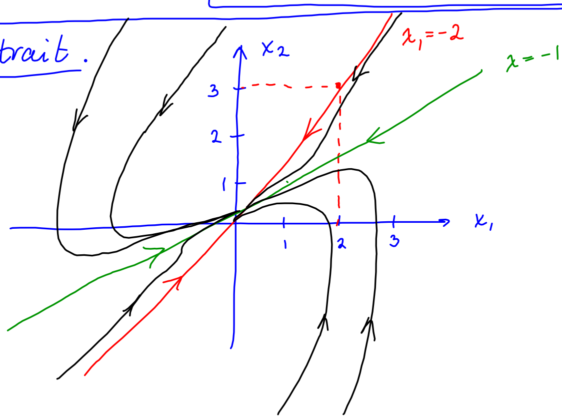
$(A + I)\vec{w} = \vec{0}$ or $\begin{pmatrix} 2 & -2 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

component form: $\begin{cases} 2w_1 - 2w_2 = 0 \\ 3w_1 - 3w_2 = 0 \end{cases}$ $\qquad w_1 = w_2$

$\vec{w} = \begin{pmatrix} w_1 \\ w_1 \end{pmatrix} = w_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\boxed{\vec{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ corresponds to } \lambda_2 = -1}$

General solution: $\boxed{\vec{x}(t) = c_1\begin{pmatrix} 2 \\ 3 \end{pmatrix}e^{-2t} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t}}$

Phase portrait.



$\lim_{t\to\infty} \vec{x}(t) = e^{-t}\left[c_1\begin{pmatrix} 2 \\ 3 \end{pmatrix}e^{-t} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]$

$= \vec{0}$

all the solutions will be parallel to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ when $t \to \infty$

$\lambda_1 < \lambda_2 < 0$

$(0,0)$ is a <u>nodal sink</u>

it is <u>asymptotically stable</u>

3. $\mathbf{x}' = \begin{pmatrix} -1 & 4 \\ -2 & 5 \end{pmatrix} \mathbf{x}$

$A = \begin{pmatrix} -1 & 4 \\ -2 & 5 \end{pmatrix}$, $\quad tr(A) = -1+5 = 4$

$\qquad\qquad\qquad\qquad det\,(A) = -5+8 = 3.$

$\lambda^2 - 4\lambda + 3 = 0$

$(\lambda-1)(\lambda-3) = 0$, $\underline{\lambda_1 = 1,\ \lambda_2 = 3}$, eigenvalues.

<u>corresponding eigenvectors</u>:

$\underline{\lambda_1 = 1}$, $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is a solution of $(A - I)\vec{v} = \vec{0}$

$\begin{pmatrix} -1-1 & 4 \\ -2 & 5-1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} -2 & 4 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

<u>component form</u>. $\quad -2v_1 + 4v_2 = 0$ or $v_1 = 2v_2$

$\vec{v} = \begin{pmatrix} 2v_2 \\ v_2 \end{pmatrix} = v_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\boxed{\vec{x}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ corresponds to } \lambda_1 = 1}$

$\underline{\lambda_2 = 3}$, $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ is a solution of $(A - 3I)\vec{w} = 0$

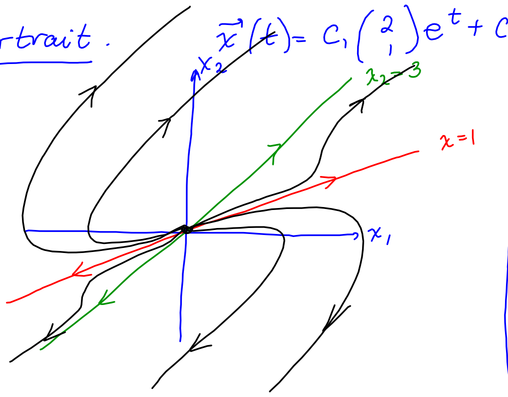
$\begin{pmatrix} -1-3 & 4 \\ -2 & 5-3 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} -4 & 4 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

component form: $\begin{cases} -4w_1 + 4w_2 = 0 \\ -2w_1 + 2w_2 = 0 \end{cases}$ or $w_1 = w_2$

$\vec{w} = \begin{pmatrix} w_1 \\ w_1 \end{pmatrix} = w_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\boxed{\vec{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ corresponds to } \lambda_2 = 3}$

<u>General solution</u>: $\boxed{\vec{x}(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}}$

<u>Phase portrait</u>. $\quad \vec{x}(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} = e^{3t}\left[c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]$



$\lambda_1 > \lambda_2 > 0$

$(0,0)$ is a <u>nodal source</u>

and it is <u>asymptotically unstable</u>.

If **A** has two distinct real eigenvalues λ_1, and λ_2, then

1. If $\lambda_1 > \lambda_2 > 0$, then the point $(0, 0)$ is a **nodal source**, and it is **asymptotically unstable**.
2. If $\lambda_1 < \lambda_2 < 0$, then the point $(0, 0)$ is a **nodal sink**, and it is **asymptotically stable**.
3. If $\lambda_1 < 0 < \lambda_2$, then the point $(0, 0)$ is a **saddle point** , and it is **unstable**.