Section 7.6 **Complex eigenvalues**

$$\mathbf{x}' = \mathbf{A}\mathbf{x},$$

here $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$.

Any complex eigenvalue must occur in conjugate pairs. If $\lambda = \alpha + i\beta$ is an eigenvalue of the matrix $\mathbf{A}$, then so is $\bar{\lambda} = \alpha - i\beta$.

If $\mathbf{v} = \mathbf{u} + i\mathbf{w}$ is an eigenvector corresponding to $\lambda = \alpha + i\beta$, then $\bar{\mathbf{v}} = \mathbf{u} - i\mathbf{w}$ is an eigenvector corresponding to $\bar{\lambda} = \alpha - i\beta$. The corresponding solution of the system is

$$\mathbf{x}(t) = (\mathbf{u} + i\mathbf{w})e^{(\alpha+i\beta)t} = (\mathbf{u} + i\mathbf{w})(\cos\beta t + i\sin\beta t)e^{\alpha t}$$

$$= (\mathbf{u}\cos\beta t - \mathbf{w}\sin\beta t))e^{\alpha t} + i\,(\mathbf{u}\sin\beta t + \mathbf{w}\cos\beta t))e^{\alpha t}$$

The vectors

$$\mathbf{x}_1(t) = \Re(\mathbf{x}(t)) = (\mathbf{u}\cos\beta t - \mathbf{w}\sin\beta t))e^{\alpha t}$$

$$\mathbf{x}_2(t) = \Im(\mathbf{x}(t)) = (\mathbf{u}\sin\beta t + \mathbf{w}\cos\beta t))e^{\alpha t}$$

are real valued solutions of the system.

Example 1. Find the general solution of the system

1. $\mathbf{x}' = \begin{pmatrix} -2 & 3 \\ -3 & -2 \end{pmatrix}\mathbf{x}$

2. $\mathbf{x}' = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix} \mathbf{x}$

3. $\mathbf{x}' = \begin{pmatrix} 0 & -1 \\ 9 & 0 \end{pmatrix} \mathbf{x}$

If $\mathbf{A}$ has two complex conjugate eigenvalues λ_1, and λ_2, then

1. If $\Re(\lambda_1) > 0$, then the point $(0, 0)$ is a **spiral source**, and it is **unstable**.
2. If $\Re(\lambda_1) < 0$, then the point $(0, 0)$ is a **spiral sink**, and it is **asymptotically stable**.
3. If $\Re(\lambda_1) = 0$, then the point $(0, 0)$ is a **center**, and it is **stable**.