Section 7.6 **Complex eigenvalues**

$$\mathbf{x}' = \mathbf{A}\mathbf{x},$$

here $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$.

Any complex eigenvalue must occur in conjugate pairs. If $\lambda = \alpha + i\beta$ is an eigenvalue of the matrix $\mathbf{A}$, then so is $\bar{\lambda} = \alpha - i\beta$.

If $\mathbf{v} = \mathbf{u} + i\mathbf{w}$ is an eigenvector corresponding to $\lambda = \alpha + i\beta$, then $\bar{\mathbf{v}} = \mathbf{u} - i\mathbf{w}$ is an eigenvector corresponding to $\bar{\lambda} = \alpha - i\beta$. The corresponding solution of the system is

$$\mathbf{x}(t) = (\mathbf{u} + i\mathbf{w})e^{(\alpha+i\beta)t} = (\mathbf{u} + i\mathbf{w})(\cos\beta t + i\sin\beta t)e^{\alpha t}$$

$$= (\mathbf{u}\cos\beta t - \mathbf{w}\sin\beta t))e^{\alpha t} + i\,(\mathbf{u}\sin\beta t + \mathbf{w}\cos\beta t))e^{\alpha t}$$

The vectors

$$\mathbf{x}_1(t) = \Re(\mathbf{x}(t)) = (\mathbf{u}\cos\beta t - \mathbf{w}\sin\beta t))e^{\alpha t}$$

$$\mathbf{x}_2(t) = \Im(\mathbf{x}(t)) = (\mathbf{u}\sin\beta t + \mathbf{w}\cos\beta t))e^{\alpha t}$$

are real valued solutions of the system.

Example 1. Find the general solution of the system

1. $\mathbf{x}' = \begin{pmatrix} -2 & 3 \\ \boxed{-3} & -2 \end{pmatrix} \mathbf{x}$

$A = \begin{pmatrix} -2 & 3 \\ -3 & -2 \end{pmatrix}$, $tr(A) = -4$, $det(A) = 4+9 = 13$

characteristic polynomial

$\lambda^2 + 4\lambda + 13 = 0.$

$\lambda_1 = \dfrac{-4+\sqrt{16-52}}{2} = \dfrac{-4+\sqrt{-36}}{2} = \dfrac{-4+6i}{2} = -2+3i$

$\lambda_2 = -2-3i = \bar{\lambda}_1$

$\lambda_1 = -2+3i$, $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is a solution of $(A-(-2+3i)I)\vec{v} = \vec{0}$

$\begin{pmatrix} -2-(-2+3i) & 3 \\ -3 & -2-(-2+3i) \end{pmatrix} \vec{v} = \vec{0}$

$\begin{pmatrix} -3i & 3 \\ -3 & 3i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -3iv_1 + 3v_2 = 0.$ $\quad v_2 = i v_1$

$\vec{v} = \begin{pmatrix} v_1 \\ i v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ i \end{pmatrix}$

$\boxed{\vec{x} = \begin{pmatrix} 1 \\ i \end{pmatrix} \text{ corresponds to } \lambda = -2+3i}$

$\vec{x} e^{\lambda t} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-2+3i)t} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-2t}(\cos 3t + i \sin 3t)$

$= e^{-2t} \begin{pmatrix} \cos 3t + i \sin 3t \\ i \cos 3t + \cancel{i^2}^{-1} \sin 3t \end{pmatrix} = e^{-2t} \begin{pmatrix} \cos 3t + i \sin 3t \\ i \cos 3t - \sin 3t \end{pmatrix}$

$= e^{-2t} \left[\begin{pmatrix} \cos 3t \\ -\sin 3t \end{pmatrix} + \begin{pmatrix} i \sin 3t \\ i \cos 3t \end{pmatrix} \right]$

$= e^{-2t} \left[\begin{pmatrix} \cos 3t \\ -\sin 3t \end{pmatrix} + i \begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix} \right]$

$\vec{x}_1 = Re(\vec{x} e^{\lambda t}) = e^{-2t} \begin{pmatrix} \cos 3t \\ -\sin 3t \end{pmatrix}$

$\vec{x}_2 = Im(\vec{x} e^{\lambda t}) = e^{-2t} \begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix}$

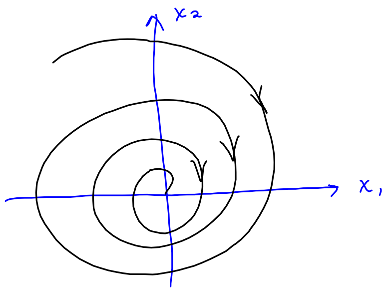
$W[\vec{x}_1, \vec{x}_2](t) = e^{-2t} \begin{vmatrix} \cos 3t & \sin 3t \\ -\sin 3t & \cos 3t \end{vmatrix} = e^{-2t} \neq 0$

General solution: $\boxed{\vec{x}(t) = e^{-2t} \left(c_1 \begin{pmatrix} \cos 3t \\ -\sin 3t \end{pmatrix} + c_2 \begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix} \right)}$

phase portrait:

$a_{21} = -3 < 0$

clockwise direction



$(0,0)$ is a spiral sink, and it is asymptotically stable

2. $\mathbf{x}' = \begin{pmatrix} 2 & 3 \\ \boxed{-3} & 2 \end{pmatrix} \mathbf{x}$, $A = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}$, $tr(A) = 4$, $det(A) = 13$

characteristic polynomial $\lambda^2 - 4\lambda + 13 = 0$

$$\lambda_1 = \frac{4 + \sqrt{16 - 52}}{2} = 2 + 3i$$

eigenvalues.

$$\lambda_2 = \overline{\lambda_1} = 2 - 3i$$

$\lambda_1 = 2+3i$, $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$,

$(A - (2+3i)I)\vec{v} = \vec{0}$ or $\begin{pmatrix} 2-(2+3i) & 3 \\ -3 & 2-(2+3i) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} -3i & 3 \\ -3 & -3i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ or $-3iv_1 + 3v_2 = 0$

$v_2 = iv_1$

$\vec{v} = \begin{pmatrix} v_1 \\ iv_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ i \end{pmatrix} \Rightarrow$ $\boxed{\vec{x} = \begin{pmatrix} 1 \\ i \end{pmatrix} \text{ corresponds to } \lambda = 2+3i}$

$$\vec{x} e^{\lambda t} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(2+3i)t} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{2t} (\cos 3t + i \sin 3t)$$

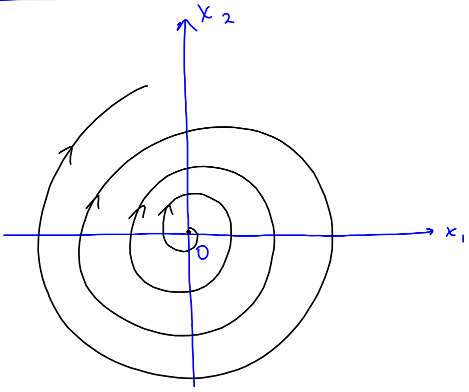
$$= e^{2t} \begin{pmatrix} \cos 3t + i \sin 3t \\ i \cos 3t + \overset{-1}{\cancel{i^2}} \sin 3t \end{pmatrix} = e^{2t} \left[\begin{pmatrix} \cos 3t \\ -\sin 3t \end{pmatrix} + i \begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix} \right]$$

$$\vec{x}_1 = Re(\vec{x} e^{\lambda t}) = e^{2t} \begin{pmatrix} \cos 3t \\ -\sin 3t \end{pmatrix}$$

$$\vec{x}_2 = Im(\vec{x} e^{\lambda t}) = e^{2t} \begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix}$$

General solution: $\boxed{\vec{x}(t) = e^{2t} \left[C_1 \begin{pmatrix} \cos 3t \\ -\sin 3t \end{pmatrix} + C_2 \begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix} \right]}$

Phase portrait



$(0,0)$ is a spiral source, and it is unstable.

3. $\mathbf{x}' = \begin{pmatrix} 0 & -1 \\ \boxed{9} & 0 \end{pmatrix} \mathbf{x}$, $A = \begin{pmatrix} 0 & -1 \\ 9 & 0 \end{pmatrix}$, $tr(A) = 0$, $det(A) = 9$

characteristic polynomial $\lambda^2 + 9 = 0$, $\lambda^2 = -9$

$\lambda = \pm 3i$ eigenvalues.

$\lambda_1 = 3i$, $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $(A - (3i)I)\vec{v} = \vec{0}$

$$\begin{pmatrix} -3i & -1 \\ 9 & -3i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} -3i\, v_1 - v_2 = 0 \\ v_2 = -3i\, v_1 \end{array}$$

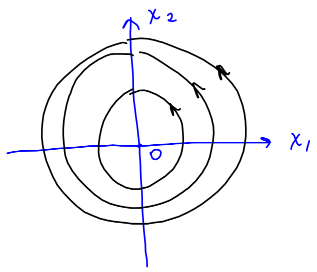
$\vec{v} = \begin{pmatrix} v_1 \\ -3i\, v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ -3i \end{pmatrix}$, $\boxed{\vec{x} = \begin{pmatrix} 1 \\ -3i \end{pmatrix} \text{ corresponds to } \lambda_1 = 3i}$

$$\vec{x} e^{\lambda t} = \begin{pmatrix} 1 \\ -3i \end{pmatrix} e^{3it} = \begin{pmatrix} 1 \\ -3i \end{pmatrix} (\cos 3t + i \sin 3t)$$

$$= \begin{pmatrix} \cos 3t + i \sin 3t \\ -3i \cos 3t - 3i^2 \sin 3t \end{pmatrix} = \begin{pmatrix} \cos 3t \\ 3 \sin 3t \end{pmatrix} + i \begin{pmatrix} \sin 3t \\ -3\cos 3t \end{pmatrix}$$

General solution $\boxed{\vec{x}(t) = C_1 \begin{pmatrix} \cos 3t \\ 3 \sin 3t \end{pmatrix} + C_2 \begin{pmatrix} \sin 3t \\ -3 \cos 3t \end{pmatrix}}$

Phase portrait.



$(0,0)$ is a center, and it is stable

If $\mathbf{A}$ has two complex conjugate eigenvalues λ_1, and λ_2, then

1. If $\Re(\lambda_1) > 0$, then the point $(0,0)$ is a **spiral source**, and it is **unstable**.
2. If $\Re(\lambda_1) < 0$, then the point $(0,0)$ is a **spiral sink**, and it is **asymptotically stable**.
3. If $\Re(\lambda_1) = 0$, then the point $(0,0)$ is a **center**, and it is **stable**.