Section 7.8 **Repeated eigenvalues**

$$\mathbf{x}' = \mathbf{A}\mathbf{x},$$

here $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{pmatrix}$.

Let λ_i be an eigenvalue of the matrix $\mathbf{A}$ of multiplicity $1 < k \leq n$. Then there are k linearly independent eigenvectors $\mathbf{v}_1(t)$,...,$\mathbf{v}_1(t)$ corresponding to λ, if $k < n$. If $k = n$, then there is only one vector $\mathbf{v}_1(t)$ corresponding to λ. The remaining $n-1$ vectors corresponding to λ are solutions to the system

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_{i+1}(t) = \mathbf{v}_i(t), \qquad i = 1, 2, ..., n-1$$

Vectors $\mathbf{v}_2(t)$, ... $\mathbf{v}_{n-1}(t)$ are called **generalized eigenvectors** corresponding to λ.

Then the corresponding solutions of the system are $\mathbf{v}_1(t)e^{\lambda t}$, $t\mathbf{v}_2(t)e^{\lambda t}$,..., $t^k\mathbf{v}_k(t)e^{\lambda t}$.

If $n = 2$ and the matrix $\mathbf{A}$ has one repeated eigenvalue λ with the corresponding eigenvector $\mathbf{v}$, then a **generalized eigenvector w** is a solution of the system

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{w} = \mathbf{v}$$

and the **general solution** of the system is

$$\mathbf{x}(t) = c_1\mathbf{v}e^{\lambda t} + c_2\left(\mathbf{v}t + \mathbf{w}\right)e^{\lambda t}$$

Example 1. Find the general solution of the system.

1. $\mathbf{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\mathbf{x}$

2. $\mathbf{x}' = \begin{pmatrix} -7 & 1 \\ -4 & -3 \end{pmatrix} \mathbf{x}$

If $\mathbf{A}$ has a repeated eigenvalue λ, then

1. If $\lambda > 0$, then the point $(0, 0)$ is an **improper node**, and it is **unstable**.
2. If $\lambda < 0$, then the point $(0, 0)$ is an **improper node**, and it is **asymptotically stable**.