Section 7.8 **Repeated eigenvalues**

$$\mathbf{x}' = \mathbf{A}\mathbf{x},$$

here $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$.

Let λ_i be an eigenvalue of the matrix $\mathbf{A}$ of multiplicity $1 < k \leq n$. Then there are k linearly independent eigenvectors $\mathbf{v}_1(t)$,...,$\mathbf{v}_1(t)$ corresponding to λ, if $k < n$. If $k = n$, then there is only one vector $\mathbf{v}_1(t)$ corresponding to λ. The remaining $n-1$ vectors corresponding to λ are solutions to the system

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_{i+1}(t) = \mathbf{v}_i(t), \qquad i = 1, 2, ..., n-1$$

Vectors $\mathbf{v}_2(t)$, ... $\mathbf{v}_{n-1}(t)$ are called **generalized eigenvectors** corresponding to λ.
Then the corresponding solutions of the system are $\mathbf{v}_1(t)e^{\lambda t}$, $t\mathbf{v}_2(t)e^{\lambda t}$,..., $t^k\mathbf{v}_k(t)e^{\lambda t}$.

If $n = 2$ and the matrix $\mathbf{A}$ has one repeated eigenvalue λ with the corresponding eigenvector $\mathbf{v}$, then a **generalized eigenvector w** is a solution of the system

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{w} = \mathbf{v}$$

and the **general solution** of the system is

$$\mathbf{x}(t) = c_1\mathbf{v}e^{\lambda t} + c_2\left(\mathbf{v}t + \mathbf{w}\right)e^{\lambda t}$$

Example 1. Find the general solution of the system.

1. $\mathbf{x}' = \begin{pmatrix} 3 & -4 \\ \boxed{1} & -1 \end{pmatrix} \mathbf{x}$, $A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$, $tr(A) = 2$, $\det(A) = 1$

characteristic polynomial is $\lambda^2 - 2\lambda + 1 = 0$

$(\lambda - 1)^2 = 0$

$\lambda = 1$ <u>repeated eigenvalue</u>

$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $(A - I)\vec{v} = \vec{0}$ or $\begin{pmatrix} 3-1 & -4 \\ 1 & -1-1 \end{pmatrix} \vec{v} = \vec{0}$

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 2v_1 - 4v_2 = 0 \\ v_1 - 2v_2 = 0 \end{cases} \text{ or } v_1 = 2v_2$$

$\vec{v} = \begin{pmatrix} 2v_2 \\ v_2 \end{pmatrix} = v_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\boxed{\vec{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ corresponds to } \lambda = 1}$

Generalized eigenvector $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ is a solution of

$$(A - I)\vec{w} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

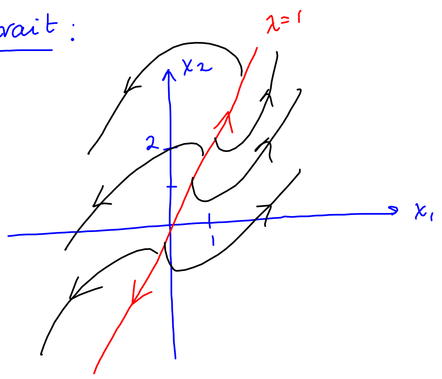
$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} 2w_1 - 4w_2 = 2 \\ w_1 - 2w_2 = 1 \end{cases} \Rightarrow w_1 = 1 + 2w_2$$

$$\vec{w} = \begin{pmatrix} 1 + 2w_2 \\ w_2 \end{pmatrix} \overset{w_2 = 0}{=} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

General solution:

$$\boxed{\vec{x}(t) = e^t \left[c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \left(t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \right]}$$

<u>Phase portrait</u>:



$\lambda > 0$

$(0,0)$ is an <u>improper node</u>,

it is <u>unstable</u>

2. $\mathbf{x}' = \begin{pmatrix} -7 & 1 \\ \boxed{-4} & -3 \end{pmatrix} \mathbf{x}$, $A = \begin{pmatrix} -7 & 1 \\ -4 & -3 \end{pmatrix}$, $tr(A) = -10$, $\det(A) = 25$

characteristic polynomial $\lambda^2 + 10\lambda + 25 = 0$

$(\lambda+5)^2 = 0$, $\underline{\lambda = -5 \text{ repeated eigenvalue}}$

$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $(A + 5I)\vec{v} = \vec{0}$

$$\begin{pmatrix} -7+5 & 1 \\ -4 & -3+5 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-2v_1 + v_2 = 0$ or $v_2 = 2v_1$

$$\vec{v} = \begin{pmatrix} v_1 \\ 2v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \boxed{\vec{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ corresponds to } \lambda = -5}$$

Generalized eigen vector: $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

$$(A + 5I)\vec{w} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow \begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

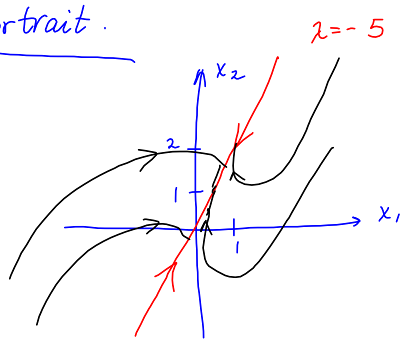
$-2w_1 + w_2 = 1 \Rightarrow w_2 = 1 + 2w_1$

$$\vec{w} = \begin{pmatrix} w_1 \\ 1+2w_1 \end{pmatrix} \overset{w_1=0}{=\!=} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

General solution

$$\boxed{\vec{x}(t) = \left[c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] e^{-5t}}$$

Phase portrait.



$\lambda < 0$

$(0,0)$ is an improper node

asymptotically stable.

If $\mathbf{A}$ has a repeated eigenvalue λ, then

1. If $\lambda > 0$, then the point $(0, 0)$ is an **improper node**, and it is **unstable**.
2. If $\lambda < 0$, then the point $(0, 0)$ is an **improper node**, and it is **asymptotically stable**.