Section 7.9 **Nonhomogeneous linear systems**

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{g}(\mathbf{t}), \tag{1}$$

here $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$, $\mathbf{g}(\mathbf{t}) = \begin{pmatrix} g_1(t) \\ g_2(t) \\ \dots \\ g_n(t) \end{pmatrix}$. We assume, that $\mathbf{A}(t)$ and $\mathbf{g}(t)$ are continuous on an open interval I.

The general solution of the system (1) is

$$\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_{\mathbf{p}}(t),$$

where $\mathbf{x}_h(t)$ is the general solution of the corresponding homogeneous system

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x}, \tag{2}$$

and $\mathbf{x}_p(t)$ is a particular solution of the nonhomogeneous system (1).

There are several methods for determining the particular solution $\mathbf{x}_p(t)$.

Undetermined coefficients. This method is applicable only if the coefficient matrix $\mathbf{A}(t)$ is a constant matrix, and if the components of $\mathbf{g}(t)$ are polynomial, exponential, or sinusoidal functions, or sums or products of these. In these cases we can choose a particular solution of the same form.

NOTE: in the case of a nonhomogeneous term of the form $\mathbf{u}e^{\lambda t}$, where λ is a simple root of the characteristic polynomial, it is necessary to use $\mathbf{x}_p(t) = \mathbf{a}te^{\lambda t} + \mathbf{b}e^{\lambda t}$, where $\mathbf{a}$ and $\mathbf{b}$ are determined by substituting into the differential system.

Example 1. Consider the nonhomogeneous linear system $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^t \\ t \end{pmatrix}$

1. Find the general solution of the corresponding homogeneous system.

2. Find the general solution of the nonhomogeneous system.

Variation of parameters.

Definition. Suppose that $\mathbf{x}^{(1)}(t)$,...,$\mathbf{x}^{(n)}(t)$ form a fundamental set of solutions for the system (2) on some interval I. Then the matrix

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} x_1^{(1)}(t) & \cdots & x_1^{(n)}(t) \\ \vdots & & \vdots \\ x_n^{(1)}(t) & \cdots & x_n^{(n)}(t) \end{pmatrix}, \tag{3}$$

whose columns are the vectors $\mathbf{x}^{(1)}(t)$,...,$\mathbf{x}^{(n)}(t)$, is said to be a **fundamental matrix** of the system (2).

Note that $\det \boldsymbol{\Psi}(t) \neq 0$.

Then the general solution of the system (2) can be written as

$$\mathbf{x} = \boldsymbol{\Psi}(t)\mathbf{c},$$

where $\mathbf{c}$ is a constant vector with arbitrary components c_1,...,c_n.

Now we replace the constant vector $\mathbf{c}$ by a vector function $\mathbf{u}(t)$. Thus, we assume that

$$\mathbf{x} = \boldsymbol{\Psi}(t)\mathbf{u}(t),$$

where $\mathbf{u}(t)$ is a vector to be found. Then we plug $\mathbf{x}(t)$ back into the system (1):

Thus,

$$\mathbf{u}(t) = \int \boldsymbol{\Psi}^{-1}(t)\mathbf{g}(t)dt + \mathbf{c}$$

and the general solution of the system (1) is

$$\mathbf{x}(t) = \boldsymbol{\Psi}(t)\mathbf{c} + \boldsymbol{\Psi}(t)\int \boldsymbol{\Psi}^{-1}(t)\mathbf{g}(t)dt$$

Example 2. Find the general solution of the system

$$\mathbf{x}' = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} t^{-1} \\ 2t^{-1} + 4 \end{pmatrix}, \qquad t > 0$$

Laplace Transform.

$$\mathcal{L}\{\mathbf{x}(t)\} = \mathbf{X}(s)$$

$$\mathcal{L}\{\mathbf{x}'(t)\} = s\mathbf{X}(s) - \mathbf{x}(0)$$

If $\mathcal{L}\{\mathbf{g}(t)\} = \mathbf{G}(s)$, then

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{G}(s).$$

Example 3. Find the general solution of the system

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}$$