Section 7.9 **Nonhomogeneous linear systems**

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{g}(\mathbf{t}), \quad (1)$$

here $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$, $\mathbf{g}(\mathbf{t}) = \begin{pmatrix} g_1(t) \\ g_2(t) \\ \dots \\ g_n(t) \end{pmatrix}$. We assume, that $\mathbf{A}(t)$ and $\mathbf{g}(t)$ are continuous on an open interval I.

The general solution of the system (1) is

$$\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_\mathbf{p}(t),$$

where $\mathbf{x}_h(t)$ is the general solution of the corresponding homogeneous system

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x}, \quad (2)$$

and $\mathbf{x}_p(t)$ is a particular solution of the nonhomogeneous system (1).

There are several methods for determining the particular solution $\mathbf{x}_p(t)$.

Undetermined coefficients. This method is applicable only if the coefficient matrix $\mathbf{A}(t)$ is a constant matrix, and if the components of $\mathbf{g}(t)$ are polynomial, exponential, or sinusoidal functions, or sums or products of these. In these cases we can choose a particular solution of the same form.

NOTE: in the case of a nonhomogeneous term of the form $\mathbf{u}e^{\lambda t}$, where λ is a simple root of the characteristic polynomial, it is necessary to use $\mathbf{x}_p(t) = \mathbf{a}te^{\lambda t} + \mathbf{b}e^{\lambda t}$, where $\mathbf{a}$ and $\mathbf{b}$ are determined by substituting into the differential system

Example 1. Consider the nonhomogeneous linear system $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} e^t \\ t \end{pmatrix}$

1. Find the general solution of the corresponding homogeneous system.

$\vec{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\vec{x}$, $A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$, $tr(A) = 0$, $\det A = -4+3 = -1$

characteristic polynomial: $\lambda^2 - 1 = 0$ or $\lambda = \pm 1$ eigenvalues

Eigen vectors. $\lambda_1 = 1$, $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$(A - I)\vec{v} = \vec{0}$ or $\begin{pmatrix} 2-1 & -1 \\ 3 & -2-1 \end{pmatrix}\vec{v} = \vec{0}$ or $\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ or $v_1 = v_2$

$\vec{v} = \begin{pmatrix} v_1 \\ v_1 \end{pmatrix} \overset{v_1=1}{=} \boxed{\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ corresponds to } \lambda = 1}$

$\lambda_2 = -1$, $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$, $(A + I)\vec{w} = \vec{0}$ or $\begin{pmatrix} 2+1 & -1 \\ 3 & -2+1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ or $3w_1 - w_2 = 0 \Rightarrow w_2 = 3w_1$

$\vec{w} = \begin{pmatrix} w_1 \\ 3w_1 \end{pmatrix} \overset{w_1=1}{=} \boxed{\begin{pmatrix} 1 \\ 3 \end{pmatrix} \text{ corresponds to } \lambda = -1}$

General solution of the homogeneous system

$$\boxed{\vec{x}_h(t) = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{t} + c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{-t}}$$

$$\vec{g}(t)=\begin{pmatrix} e^t \\ t \end{pmatrix}=\begin{pmatrix} e^t \\ 0 \end{pmatrix}+\begin{pmatrix} 0 \\ t \end{pmatrix}$$

$\lambda=1$ is an eigenvalue

2. Find the general solution of the nonhomogeneous system.

$$\vec{x}_p(t)=\underbrace{\vec{a}e^t+\vec{b}te^t}_{\begin{pmatrix} e^t \\ 0 \end{pmatrix}}+\underbrace{\vec{c}t+\vec{d}}_{\begin{pmatrix} 0 \\ t \end{pmatrix}}$$

$$\begin{pmatrix} x_{p1}(t) \\ x_{p2}(t) \end{pmatrix}=\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}e^t+\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}te^t+\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}t+\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

$$=\begin{pmatrix} a_1e^t+b_1te^t+c_1t+d_1 \\ a_2e^t+b_2te^t+c_2t+d_2 \end{pmatrix}$$

$$\begin{pmatrix} x_{p1}(t) \\ x_{p2}(t) \end{pmatrix}'=\begin{pmatrix} a_1e^t+b_1e^t+b_1te^t+c_1 \\ a_2e^t+b_2e^t+b_2te^t+c_2 \end{pmatrix}$$

$$\begin{pmatrix} a_1e^t+b_1e^t+b_1te^t+c_1 \\ a_2e^t+b_2e^t+b_2te^t+c_2 \end{pmatrix}=\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} a_1e^t+b_1te^t+c_1t+d_1 \\ a_2e^t+b_2te^t+c_2t+d_2 \end{pmatrix}+\begin{pmatrix} e^t \\ t \end{pmatrix}$$

$$=\begin{pmatrix} 2a_1e^t+2b_1te^t+2c_1t+2d_1-a_2e^t-b_2te^t-c_2t-d_2+e^t \\ 3a_1e^t+3b_1te^t+3c_1t+3d_1-2a_2e^t-2b_2te^t-2c_2t-2d_2+t \end{pmatrix}$$

1st component

e^t: $a_1+b_1=2a_1-a_2+1$

te^t: $b_1=2b_1-b_2$

t: $0=2c_1-c_2$

1: $c_1=2d_1-d_2$

$$\begin{cases} c_2=2c_1 \\ 3c_1-2c_2+1=0 \end{cases}$$

$3c_1-4c_1+1=0$

$-c_1+1=0 \Rightarrow \boxed{c_1=1}\ \boxed{c_2=2}$

2nd component

e^t: $a_2+b_2=3a_1-2a_2$

te^t: $b_2=3b_1-2b_2$

t: $0=3c_1-2c_2+1$

1: $c_2=3d_1-2d_2$

$$\begin{cases} 1=2d_1-d_2 \Rightarrow d_2=2d_1-1 \\ 2=3d_1-2d_2 \end{cases}$$

$\cancel{2}=3d_1-4d_1+\cancel{2} \Rightarrow \boxed{d_1=0}\ \boxed{d_2=-1}$

$$\begin{cases} a_1+b_1=2a_1-a_2+1 \\ b_1=2b_1-b_2 \\ a_2+b_2=3a_1-2a_2 \\ b_2=3b_1-2b_2 \end{cases}$$

$b_1=b_2$

$$\begin{cases} a_1-a_2-b_1=-1 \\ 3a_1-3a_2-b_2=0 \end{cases}$$

$b_1=b_2=3a_1-3a_2$

$a_1-a_2-3(a_1-a_2)=-1$

$-2a_1+2a_2=-1 \Rightarrow a_1-a_2=\frac{1}{2}$

$a_1=\dfrac{-1-2a_2}{-2}=\frac{1}{2}+a_2$ plug $\boxed{a_2=0,\ a_1=1/2}$

$\boxed{b_1=b_2=\frac{3}{2}}$

$$\boxed{\vec{x}_p(t)=\begin{pmatrix} 1/2 \\ 0 \end{pmatrix}e^t+\begin{pmatrix} 3/2 \\ 3/2 \end{pmatrix}te^t+\begin{pmatrix} 1 \\ 2 \end{pmatrix}t+\begin{pmatrix} 0 \\ -1 \end{pmatrix}}$$

General solution of the nonhomogeneous system

$$\vec{x}(t)=\vec{x}_h(t)+\vec{x}_p(t)$$

$$\boxed{\vec{x}(t)=c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^t+c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{-t}+\begin{pmatrix} 1/2 \\ 0 \end{pmatrix}e^t+\begin{pmatrix} 3/2 \\ 3/2 \end{pmatrix}te^t+\begin{pmatrix} 1 \\ 2 \end{pmatrix}t+\begin{pmatrix} 0 \\ -1 \end{pmatrix}}$$

$\vec{x}' = A(t)\vec{x} + \vec{g}$ (1)

$\vec{x}' = A(t)\vec{x}$ (2)

Variation of parameters.

Definition. Suppose that $\mathbf{x}^{(1)}(t),...,\mathbf{x}^{(n)}(t)$ form a fundamental set of solutions for the system (2) on some interval I. Then the matrix

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} x_1^{(1)}(t) & \cdots & x_1^{(n)}(t) \\ \vdots & & \vdots \\ x_n^{(1)}(t) & \cdots & x_n^{(n)}(t) \end{pmatrix}, \tag{3}$$

whose columns are the vectors $\mathbf{x}^{(1)}(t),...,\mathbf{x}^{(n)}(t)$, is said to be a **fundamental matrix** of the system (2).

Note that $\det \boldsymbol{\Psi}(t) \neq 0$.
Then the general solution of the system (2) can be written as

$$\mathbf{x} = \boldsymbol{\Psi}(t)\mathbf{c},$$

where $\mathbf{c}$ is a constant vector with arbitrary components $c_1,...,c_n$.
Now we replace the constant vector $\mathbf{c}$ by a vector function $\mathbf{u}(t)$. Thus, we assume that

$$\mathbf{x} = \boldsymbol{\Psi}(t)\mathbf{u}(t),$$

where $\mathbf{u}(t)$ is a vector to be found. Then we plug $\mathbf{x}(t)$ back into the system (1):

$\vec{x}'(t) = \Psi'(t)\vec{u}(t) + \Psi(t)\vec{u}'(t)$

$\cancel{\Psi'(t)\vec{u}(t)} + \Psi(t)\vec{u}'(t) = \cancel{A(t)\Psi(t)\vec{u}(t)} + \vec{g}(t)$

since $\Psi(t)$ is the fundamental matrix of (2),
then $\Psi'(t) = A(t)\Psi(t)$

$\Psi(t)\,\vec{u}'(t) = \vec{g}(t)$

$\vec{u}'(t) = \Psi^{-1}(t)\vec{g}(t)$

Thus,

$$\mathbf{u}(t) = \int \boldsymbol{\Psi}^{-1}(t)\mathbf{g}(t)dt + \mathbf{c}$$

and the general solution of the system (1) is

$$\boxed{\mathbf{x}(t) = \boldsymbol{\Psi}(t)\mathbf{c} + \boldsymbol{\Psi}(t)\int \boldsymbol{\Psi}^{-1}(t)\mathbf{g}(t)dt}$$

Example 2. Find the general solution of the system

$$\mathbf{x}' = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} t^{-1} \\ 2t^{-1}+4 \end{pmatrix}, \qquad t > 0$$

Homogeneous system: $\vec{x}' = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \vec{x}$, $A = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}$, $tr(A) = -5$, $det(A) = 0$

characteristic polynomial: $\lambda^2 + 5\lambda = 0$

$\lambda(\lambda+5) = 0$

$\lambda_1 = 0, \quad \lambda_2 = -5$ – eigenvalues

$\underline{\lambda_1 = 0.}$ $\quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $\quad (A - 0\cdot I)\vec{v} = \vec{0}$, $\quad A\vec{v} = \vec{0}$

$$\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or} \quad 2v_1 - v_2 = 0 \Rightarrow v_2 = 2v_1$$

$$\vec{v} = \begin{pmatrix} v_1 \\ 2v_1 \end{pmatrix} \underset{=}{\overset{v_1=1}{}} \boxed{\begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ corresponds to } \lambda = 0}$$

$\lambda_2 = -5.$ $\quad \vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$, $\quad (A - (-5)I)\vec{w} = \vec{0}$ or $\begin{pmatrix} -4+5 & 2 \\ 2 & -1+5 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or} \quad w_1 + 2w_2 = 0 \quad \text{or} \quad w_1 = -2w_2$$

$$\vec{w} = \begin{pmatrix} -2w_2 \\ w_2 \end{pmatrix} \underset{=}{\overset{w_2=1}{}} \boxed{\begin{pmatrix} -2 \\ 1 \end{pmatrix} \text{ corresponds to } \lambda = -5}$$

General solution of the homogeneous system is:

$$\vec{x}_p(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{0\cdot t} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-5t}$$

$$= c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-5t}$$

Fundamental matrix of the system

$$\Psi(t) = \begin{pmatrix} 1 & -2e^{-5t} \\ 2 & e^{-5t} \end{pmatrix}$$

Fundamental matrix of the system

$$\Psi(t)=\begin{pmatrix}1 & -2e^{-5t}\\ 2 & e^{-5t}\end{pmatrix},\qquad \det\Psi(t)=e^{-5t}+4e^{-5t}=5e^{-5t}$$

$$\vec{x}(t)=\Psi(t)\vec{c}+\underbrace{\Psi(t)\int\Psi(t)^{-1}\vec{g}(t)\,dt}$$

$$\Psi^{-1}(t)=\frac{1}{5e^{-5t}}\begin{pmatrix}e^{-5t} & 2e^{-5t}\\ -2 & 1\end{pmatrix}=\begin{pmatrix}\frac{1}{5} & \frac{2}{5}\\ -\frac{2}{5}e^{5t} & \frac{1}{5}e^{5t}\end{pmatrix}$$

$$\Psi^{-1}(t)\vec{g}(t)=\begin{pmatrix}\frac{1}{5} & \frac{2}{5}\\ -\frac{2}{5}e^{5t} & \frac{1}{5}e^{5t}\end{pmatrix}\begin{pmatrix}\frac{1}{t}\\ \frac{2}{t}+4\end{pmatrix}=\begin{pmatrix}\frac{1}{5t}+\frac{2}{5}\left(\frac{2}{t}+4\right)\\ -\frac{2}{5}e^{5t}\frac{1}{t}+\frac{1}{5}e^{5t}\left(\frac{2}{t}+4\right)\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{5t}+\frac{4}{5t}+\frac{8}{5}\\ -\frac{2}{5}e^{5t}\frac{1}{t}+\frac{2}{5}e^{5t}\frac{1}{t}+\frac{4}{5}e^{5t}\end{pmatrix}=\begin{pmatrix}\frac{1}{t}+\frac{8}{5}\\ \frac{4}{5}e^{5t}\end{pmatrix}$$

$$\int\begin{pmatrix}\frac{1}{t}+\frac{8}{5}\\ \frac{4}{5}e^{5t}\end{pmatrix}dt=\begin{pmatrix}\int\left(\frac{1}{t}+\frac{8}{5}\right)dt\\ \int\frac{4}{5}e^{5t}\,dt\end{pmatrix}=\begin{pmatrix}-\frac{1}{t^2}+\frac{8}{5}t+\cancel{c_1}\,0\\ \frac{4}{25}e^{5t}+\cancel{c_2}\,0\end{pmatrix}$$

$$\vec{x}_p(t)=\begin{pmatrix}1 & -2e^{-5t}\\ 2 & e^{-5t}\end{pmatrix}\begin{pmatrix}-\frac{1}{t^2}+\frac{8}{5}t\\ \frac{4}{25}e^{5t}\end{pmatrix}=\begin{pmatrix}-\frac{1}{t^2}+\frac{8}{5}t-\frac{8}{25}e^{-5t}e^{5t}\\ -\frac{2}{t^2}+\frac{16}{5}t+\frac{4}{25}e^{-5t}e^{5t}\end{pmatrix}$$

$$\vec{x}_p(t)=\begin{pmatrix}-\frac{1}{t^2}+\frac{8}{5}t-\frac{8}{25}\\ -\frac{2}{t^2}+\frac{16}{5}t+\frac{4}{25}\end{pmatrix}$$

General solution of the nonhomogeneous system:

$$\boxed{\vec{x}(t)=\underbrace{c_1\begin{pmatrix}1\\2\end{pmatrix}+c_2\begin{pmatrix}-2\\1\end{pmatrix}e^{-5t}}_{\Psi(t)\vec{c}}-\begin{pmatrix}1\\2\end{pmatrix}\frac{1}{t^2}+\begin{pmatrix}1\\2\end{pmatrix}\frac{8}{5}t+\begin{pmatrix}-2\\1\end{pmatrix}\frac{4}{25}}$$

Laplace Transform.

$$\mathcal{L}\{\mathbf{x}(t)\} = \mathbf{X}(s) = \begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix}$$

$$\mathcal{L}\{\mathbf{x}'(t)\} = s\mathbf{X}(s) - \mathbf{x}(0)$$

If $\mathcal{L}\{\mathbf{g}(t)\} = \mathbf{G}(s)$, then

$$\boxed{\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{G}(s).} \qquad \vec{x}(t) = \mathcal{L}^{-1}\{\vec{X}(s)\}$$

Example 3. Find the general solution of the system

$$\mathcal{L}\left\{\mathbf{x}'\right\} = \mathcal{L}\left\{\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}\right\} \qquad \text{assume that } \vec{x}(0) = \vec{0}$$

$$\mathcal{L}\{\vec{x}(t)\} = \vec{X}(s)$$

$$\mathcal{L}\{\vec{x}'(t)\} = s\vec{X}(s) - \cancel{\vec{x}(0)}^{\,0}$$

$$s\vec{X}(s) = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\vec{X}(s) + \begin{pmatrix} \frac{1}{s+2} \\ -\frac{2}{s-1} \end{pmatrix}$$

$$s\vec{X}(s) - \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\vec{X}(s) = \begin{pmatrix} \frac{1}{s+2} \\ -\frac{2}{s-1} \end{pmatrix}$$

$$\left[sI - \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\right]\vec{x}(s) = \begin{pmatrix} \frac{1}{s+2} \\ -\frac{2}{s-1} \end{pmatrix}$$

$$\begin{pmatrix} s-1 & -1 \\ -4 & s+2 \end{pmatrix}\vec{X}(s) = \begin{pmatrix} \frac{1}{s+2} \\ -\frac{2}{s-1} \end{pmatrix}$$

$$\vec{X}(s) = \begin{pmatrix} s-1 & -1 \\ -4 & s+2 \end{pmatrix}^{-1}\begin{pmatrix} \frac{1}{s+2} \\ -\frac{2}{s-1} \end{pmatrix}$$

$$\det(sI - A) = (s-1)(s+2) - 4 = s^2 + s - 2 - 4 = s^2 + s - 6$$

$$(sI - A)^{-1} = \frac{1}{s^2+s-6}\begin{pmatrix} s+2 & 1 \\ 4 & s-1 \end{pmatrix} = \begin{pmatrix} \frac{s+2}{s^2+s-6} & \frac{1}{s^2+s-6} \\ \frac{4}{s^2+s-6} & \frac{s-1}{s^2+s-6} \end{pmatrix}$$

$$\vec{X}(s) = \begin{pmatrix} \frac{s+2}{s^2+s-6} & \frac{1}{s^2+s-6} \\ \frac{4}{s^2+s-6} & \frac{s-1}{s^2+s-6} \end{pmatrix} \begin{pmatrix} \frac{1}{s+2} \\ -\frac{2}{s-1} \end{pmatrix} = \begin{pmatrix} \frac{\cancel{s+2}}{s^2+s-6} \frac{1}{\cancel{s+2}} + \frac{1}{s^2+s-6}\left(-\frac{2}{s-1}\right) \\ \frac{4}{s^2+s-6}\frac{1}{s+2} - \frac{2}{\cancel{s-1}} \frac{\cancel{s-1}}{s^2+s-6} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{s^2+s-6} - \frac{2}{(s^2+s-6)(s-1)} \\ \frac{4}{(s^2+s-6)(s+2)} - \frac{2}{s^2+s-6} \end{pmatrix} = \begin{pmatrix} \frac{s-3}{(s^2+s-6)(s-1)} \\ \frac{-2s}{(s^2+s-6)(s+2)} \end{pmatrix} = \begin{pmatrix} -\frac{3}{10}\frac{1}{s+3} - \frac{1}{5}\frac{1}{s-2} + \frac{1}{2}\frac{1}{s-1} \\ \frac{6}{5}\frac{1}{s+3} - \frac{1}{5}\frac{1}{s-2} - \frac{1}{s+2} \end{pmatrix}$$

Partial fractions:

$$\frac{s-3}{(s^2+s-6)(s-1)} = \frac{s-3}{(s+3)(s-2)(s-1)} = \frac{A}{s+3} + \frac{B}{s-2} + \frac{C}{s-1}$$

$$s-3 = A(s-2)(s-1) + B(s+3)(s-1) + C(s+3)(s-2)$$

$s=2$: $\quad -1 = 5B \Rightarrow \boxed{B = -\frac{1}{5}}$

$s=1$: $\quad -2 = -4C \Rightarrow \boxed{C = \frac{1}{2}}$

$s=-3$: $\quad -6 = A(-5)(-4) \Rightarrow \boxed{A = -\frac{3}{10}}$

$$\frac{-2s}{(s^2+s-6)(s+2)} = \frac{-2s}{(s+3)(s-2)(s+2)} = \frac{A}{s+3} + \frac{B}{s-2} + \frac{C}{s+2}$$

$$-2s = A(s-2)(s+2) + B(s+3)(s+2) + C(s-2)(s+3)$$

$s=2$: $\quad -4 = 20B \Rightarrow \boxed{B = -\frac{1}{5}}$

$s=-2$: $\quad 4 = -4C \Rightarrow \boxed{C = -1}$

$s=-3$: $\quad 6 = 5A \Rightarrow \boxed{A = \frac{6}{5}}$

$$\vec{X}(s) = \begin{pmatrix} -\frac{3}{10}\frac{1}{s+3} - \frac{1}{5}\frac{1}{s-2} + \frac{1}{2}\frac{1}{s-1} \\ \frac{6}{5}\frac{1}{s+3} - \frac{1}{5}\frac{1}{s-2} - \frac{1}{s+2} \end{pmatrix}$$

$$\boxed{\vec{x}(t) = \mathcal{L}^{-1}\vec{X}(s) = \begin{pmatrix} -\frac{3}{10}e^{-3t} - \frac{1}{5}e^{2t} + \frac{1}{2}e^{t} \\ \frac{6}{5}e^{-3t} - \frac{1}{5}e^{2t} - e^{-2t} \end{pmatrix}}$$