MATH 308 – 201 **Practice Problems for Exam 2** Summer 2018

over sections 3.5 – 3.8, 6.1 – 6.6, 7.1 – 7.3

1. Find the general solution of the equation/solve the initial value problem

(a) $y'' + 6y' + 9y = t\cos(2t)$

homogeneous equation: $y''+6y'+9y=0$

auxiliary equation: $r^2+6r+9=0$

$(r+3)^2=0 \Rightarrow r=-3$ - repeated root

general solution of the homogeneous eqn is $y_h(t)=(C_1+C_2t)e^{-3t}$

particular solution:

$\beta=2i$ not a root of the auxiliary equation.

$y_p=(At+B)\cos 2t+(Ct+D)\sin 2t$

$y_p'=A\cos 2t+C\sin 2t-2(At+B)\sin 2t+2(Ct+D)\cos 2t$

$y_p''=-2A\sin 2t+2C\cos 2t-2A\sin 2t+2C\cos 2t-4(At+B)\cos 2t-4(Ct+D)\sin 2t$

$=-4A\sin 2t+4C\cos 2t-4(At+B)\cos 2t-4(Ct+D)\sin 2t$

$-4A\sin 2t+4C\cos 2t-4(At+B)\cos 2t-4(Ct+D)\sin 2t+6A\cos 2t+6C\sin 2t-12(At+B)\sin 2t+12(Ct+D)\cos 2t$
$+9(At+B)\cos 2t+9(Ct+D)\sin 2t=t\cos 2t$

$t\cos 2t:$ $-4A+12C+9A=1 \Rightarrow 5A+12C=1 \Rightarrow \frac{25C}{12}+12C=1 \Rightarrow \frac{169C}{12}=1 \Rightarrow \boxed{C=\frac{12}{169}}$

$t\sin 2t:$ $-4C-12A+9C=0 \Rightarrow 5C-12A=0 \Rightarrow A=\frac{5C}{12} \Rightarrow \boxed{A=\frac{5}{169}}$

$\cos 2t:$ $4C-4B+6A+12D+9B=0$

$\sin 2t:$ $-4A-4D+6C-12B+9D=0$

$5B+12D=-6A-4C=-\frac{78}{169}$

$5D-12B=-6C+4A=-\frac{52}{169}$

$\begin{cases} 5B+12D=-\frac{78}{169} \\ 5D-12B=-\frac{52}{169} \end{cases} \Rightarrow D=\left(-\frac{52}{169}+12B\right)\frac{1}{5}$

$5B+12\left(-\frac{52}{169}+12B\right)\frac{1}{5}=-\frac{78}{169}$

$25B-\frac{624}{169}+144B=-\frac{390}{169}$

$169B=\frac{234}{169}=\frac{18}{13} \Rightarrow B=\frac{18}{13^3}=\boxed{\frac{18}{2197}=B}$

$D=\left(-\frac{52}{169}+\frac{12\cdot 18}{2197}\right)\frac{1}{5}=\boxed{-\frac{92}{2197}=D}$

$y_p(t)=\left(\frac{5}{169}t+\frac{18}{2197}\right)\cos 2t+\left(\frac{12}{169}t-\frac{92}{2197}\right)\sin 2t$

general solution of the nonhomogeneous eqn:

$$\boxed{y(t)=y_h(t)+y_p(t)=(C_1+C_2t)e^{-3t}+\left(\frac{5}{169}t+\frac{18}{2197}\right)\cos 2t+\left(\frac{12}{169}t-\frac{92}{2197}\right)\sin 2t}$$

(b) $4y'' + y' = (4t^3 + 48t^2 + 1)e^{0\cdot t}$

homogeneous equation: $4y'' + y' = 0$

auxiliary equation: $4r^2 + r = 0$

$r(4r+1) = 0 \Rightarrow r_1 = 0,\ r_2 = -\frac{1}{4}$

general solution of the homogeneous eqn. $y_h(t) = C_1 e^{0t} + C_2 e^{-\frac{t}{4}}$

$y_h(t) = C_1 + C_2 e^{-\frac{t}{4}}$

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check if $\alpha = 0$ is a root of the auxiliary eqn.

$\boxed{\text{Yes}}$.

$y_p(t) = (At^3 + Bt^2 + Ct + D)t$

$y_p(t) = At^4 + Bt^3 + Ct^2 + Dt$

$y_p' = 4At^3 + 3Bt^2 + 2Ct + D$

$y_p'' = 12At^2 + 6Bt + 2C$

$48At^2 + 24Bt + 8C + 4At^3 + 3Bt^2 + 2Ct + D = 4t^3 + 48t^2 + 1$

$t^3$: $4A = 4 \Rightarrow \boxed{A = 1}$

$t^2$: $48A + 3B = 48 \Rightarrow \boxed{B = 0}$

$t$: $24B + 2C = 0 \Rightarrow \boxed{C = 0}$

$1$: $D + 8C = 1 \Rightarrow \boxed{D = 1}$

$y_p(t) = t^4 + t$

General solution of the nonhomogeneous eqn. is $\boxed{y(t) = C_1 + C_2 e^{-\frac{t}{4}} + t^4 + t}$

(c) $y'' + 2y' + y = 4e^{-t}$, $y(0) = 2$, $y'(0) = 1$

$y''+2y'+y=0 \Rightarrow r^2+2r+1=0 \Rightarrow (r+1)^2=0 \Rightarrow r=-1$ repeated root

$y_h(t) = (c_1+c_2t)e^{-t}$

$y_p(t) = At^2e^{-t}$

$r=-1$ is the repeated root of the auxiliary equation.

$y_p' = 2Ate^{-t} - At^2e^{-t}$

$y_p'' = 2Ae^{-t} - 2Ate^{-t} - 2Ate^{-t} + At^2e^{-t}$

$= 2Ae^{-t} - 4Ate^{-t} + At^2e^{-t}$

$2Ae^{-t} - \cancel{4Ate^{-t}} + \cancel{At^2e^{-t}} + \cancel{4Ate^{-t}} - \cancel{2At^2e^{-t}} + \cancel{At^2e^{-t}} = 4e^{-t}$

$2Ae^{-t} = 4e^{-t} \Rightarrow \boxed{A=2}$

$y_p = 2t^2e^{-t}$

$\boxed{y(t) = (c_1+c_2t)e^{-t} + 2t^2e^{-t}}$

2. Find the general solution of the equation $y'' + 6y' + 9y = \frac{e^{-3x}}{1+2x}$

Variation of parameters.

$y''+6y'+9y=0$, $\quad r^2+6r+9=0$, $\quad (r+3)^2=0$, $\quad r=-3$ - repeated root

$$y_h(x) = (C_1 + C_2 x)e^{-3x} = C_1 \underbrace{e^{-3x}}_{y_1(x)} + C_2 \underbrace{xe^{-3x}}_{y_2(x)}$$

$y_1(x) = e^{-3x}$, $\quad y_2(x) = xe^{-3x}$

$$W[y_1, y_2] = \begin{vmatrix} e^{-3x} & xe^{-3x} \\ -3e^{-3x} & e^{-3x} - 3xe^{-3x} \end{vmatrix} = e^{-3x}\left(e^{-3x} - \cancel{3xe^{-3x}}\right) + \cancel{3xe^{-3x}\cdot e^{-3x}} = e^{-6x}$$

$$C_1(x) = -\int \frac{y_2(x)g(x)}{W[y_1,y_2]}dx = -\int \frac{x\cancel{e^{-3x}}}{\cancel{e^{-6x}}} \cdot \frac{\cancel{e^{-3x}}}{1+2x}dx = -\int \frac{x\,dx}{1+2x} \quad \left|\begin{array}{l} u = 1+2x \\ x = \frac{u-1}{2} \\ dx = \frac{du}{2} \end{array}\right|$$

$$= -\int \frac{\frac{u-1}{2} \cdot \frac{du}{2}}{u} = -\frac{1}{4}\int (u-1)u^{-1}du = -\frac{1}{4}\int\left(1 - \frac{1}{u}\right)du$$

$$= -\frac{1}{4}\left(u - \ln|u|\right) + C_3 = -\frac{1}{4}\left(1+2x - \ln|1+2x|\right) + C_3$$

$$C_2(x) = \int \frac{y_1(x)g(x)}{W[y_1,y_2]}dx = \int \frac{\cancel{e^{-3x}}}{\cancel{e^{-6x}}} \cdot \frac{\cancel{e^{-3x}}}{1+2x}dx = \int \frac{dx}{1+2x} = \ln|1+2x| + C_4$$

General solution:

$$\boxed{y(x) = -\frac{1}{4}\left(1+2x - \ln|1+2x|\right)e^{-3x} + C_3 e^{-3x} + xe^{-3x}\ln|1+2x| + C_4 xe^{-3x}}$$

$1+2x$

3. A spring is stretch 10 cm by a force of 3 N. A mass of 2 kg is hung from the spring and is also attached to a viscous damper that exerts a force of 3 N when the velocity of the mass 5 m/s. If the mass is pulled down 5 cm below its equilibrium position and given an initial velocity of 10 cm/s, determine its position $u$ at any time. Find the quasifrequency of the motion.

cm $\to$ m (always!)

$3 = 0.1k \Rightarrow \boxed{k=30}$

$m=2$

damping force: $3 = 5\gamma$

$\gamma = \frac{3}{5}$

$u(0) = 0.05$

$u'(0) = 0.1$

$mu'' + \gamma u' + ku = 0$

$2u'' + \frac{3}{5}u' + 30u = 0$

$10u'' + 3u' + 150u = 0, \quad u(0) = 0.05, \quad u'(0) = 0.1$

solve the initial value problem.

auxiliary equation: $10r^2 + 3r + 150 = 0$

$r_1 = \frac{-3+\sqrt{9-4(10)(150)}}{20} = \frac{-3+\sqrt{-5991}}{20} = -\frac{3}{20} + \frac{i\sqrt{5991}}{20}$

$r_2 = \overline{r_1}$

General solution: $u(t) = \left(C_1 \cos\frac{\sqrt{5991}}{20}t + C_2 \sin\frac{\sqrt{5991}}{20}t\right)e^{-\frac{3}{20}t}$

$\boxed{\text{quasifrequency} = \frac{\sqrt{5991}}{20}}$

Plug $u(t)$ into the initial conditions: $u(0) = \boxed{C_1 = 0.05}$

$u'(t) = \left(-C_1\frac{\sqrt{5991}}{20}\sin\frac{\sqrt{5991}}{20}t + C_2\frac{\sqrt{5991}}{20}\cos\frac{\sqrt{5991}}{20}t\right)e^{-\frac{3}{20}t}$

$- \frac{3}{20}\left(C_1\cos\frac{\sqrt{5991}}{20}t + C_2\sin\frac{\sqrt{5991}}{20}t\right)e^{-\frac{3}{20}t}$

$u'(0) = C_2\frac{\sqrt{5991}}{20} - C_1\frac{3}{20} = 0.1$

$C_2\sqrt{5991} - 3C_1 = 2 \Rightarrow \boxed{C_2 = \frac{2+3C_1}{\sqrt{5991}} = \frac{2.15}{\sqrt{5991}}}$

$$\boxed{u(t) = \left(0.05\cos\frac{\sqrt{5991}}{20}t + \frac{2.15}{\sqrt{5991}}\sin\frac{\sqrt{5991}}{20}t\right)e^{-\frac{3}{20}t}}$$

$m = \frac{8}{32} = \frac{1}{4}$

4. A mass weighting 8 lb is attached to a spring hanging from the ceiling and comes to rest at its equilibrium position. At $t = 0$, an external force $F(t) = 2\cos 2t$ lb is applied to the system. If the spring constant is 10 lb/ft and the damping constant is 1 lb-sec/ft, find the steady-state solution for the system. What is the resonance frequency for the system?

$$mu'' + \gamma u' + ku = 2\cos 2t$$

$$\frac{8}{32}u'' + u' + 10u = 2\cos 2t$$

$$u'' + 4u' + 40u = 8\cos 2t$$

steady-state solution $= u_p(t)$ particular solution

$$u_p(t) = A\cos 2t + B\sin 2t$$

$$u_p'(t) = -2A\sin 2t + 2B\cos 2t$$

$$u_p''(t) = -4A\cos 2t - 4B\sin 2t$$

$$-4A\cos 2t - 4B\sin 2t - 8A\sin 2t + 8B\cos 2t + 40A\cos 2t + 40B\sin 2t = 8\cos 2t$$

$\cos 2t$ : $36A + 8B = 8$

$\sin 2t$ : $36B - 8A = 0 \Rightarrow A = \frac{36B}{8} = \frac{9B}{2}$

$$36\left(\frac{9B}{2}\right) + 8B = 8$$

$$\frac{18(9B) + 8B = 8}{2 \qquad\qquad 2} \Longrightarrow 81B + 4B = 4$$

$$85B = 4 \Rightarrow \boxed{B = \frac{4}{85}}, \boxed{A = \frac{18}{85}}$$

steady-state solution: $\boxed{u(t) = \frac{18}{85}\cos 2t + \frac{4}{85}\sin 2t}$

resonance frequency : $\frac{\omega_r}{2\pi} = \frac{\sqrt{\frac{k}{m} - \frac{\gamma^2}{2m^2}}}{2\pi} = \ldots$

5. A mass weighing 4 lb stretches a spring 1.5 in. The mass is given a positive displacement 2 in from its equilibrium position and released with no initial velocity. Assuming that there is no damping and the mass is acted on by an external force of $2\cos 3t$ lb,

(a) Formulate the initial value problem describing the motion of mass

(b) Solve the initial value problem.

(c) If the given external force is replaced by a force $4\cos\omega t$ of frequency $\omega$, find the value of $\omega$ for which resonance occurs.

$m = \frac{4}{32} = \frac{1}{8}$, $\quad 4 = k \cdot \frac{1.5}{12} \Rightarrow k = \frac{48}{1.5} = 32$, $\quad \gamma = 0$

$\frac{1}{8}u'' + 32u = 2\cos 3t$

$u'' + 256u = 16\cos 3t$, $\quad u(0) = \frac{2}{12} = \frac{1}{6}$, $\quad u'(0) = 0$

transient part $\quad u_h(t) = C_1\cos 16t + C_2\sin 16t$

steady-state solution $\quad u_p(t) = A\cos 3t + B\sin 3t$

$B = 0$, $\quad A = \frac{16}{247}$

$$\boxed{\begin{aligned} &u(t) = C_1\cos 16t + C_2\sin 16t + \frac{16}{247}\cos 3t \\ &C_1 = \frac{1}{6} - \frac{16}{247} \\ &C_2 = 0 \end{aligned}}$$

$F = 4\cos\omega t$, $\quad \boxed{\omega = 16}$

6. Find the Laplace transform of the given function.

(a) $f(t) = \begin{cases} \frac{t}{2}, & 0 \le t < 6 \\ 3, & t \ge 6 \end{cases}$, $f(t) = \frac{t}{2} + \left(3 - \frac{t}{2}\right) u_6(t)$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{\frac{t}{2} + \left(3 - \frac{t}{2}\right) u_6(t)\right\}$$

$$= \mathcal{L}\left\{\frac{t}{2}\right\} + \mathcal{L}\left\{\frac{1}{2}(6-t) u_6(t)\right\}$$

$$= \frac{1}{2s^2} + \frac{1}{2}(-1)\, \mathcal{L}\{(t-6) u_6(t)\}$$

$$= \frac{1}{2s^2} - \frac{1}{2} e^{-6s} \mathcal{L}\{t\} = \boxed{\frac{1}{2s^2} - \frac{e^{-6s}}{2s^2}}$$

(b) $f(t) = (t^2 - 2t + 2)u_1(t)$

$$f(t) = [(t-1)^2 + 1]\, u_1(t)$$

$$= (t-1)^2 u_1(t) + u_1(t)$$

$$\mathcal{L}\{(t-1)^2 u_1(t) + u_1(t)\}$$

$$= e^{-s} \mathcal{L}\{t^2\} + \frac{e^{-s}}{s}$$

$$= \boxed{e^{-s} \frac{2}{s^3} + \frac{e^{-s}}{s}}$$

(c) $f(t) = \int_0^t (t-\tau)^2 \cos 2\tau \, d\tau = (g * h)(t)$

$g(t-\tau) = (t-\tau)^2 \Rightarrow g(t) = t^2$

$h(\tau) = \cos 2\tau \Rightarrow h(t) = \cos 2t$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\} \cdot \mathcal{L}\{h(t)\}$

$= \mathcal{L}\{t^2\} \cdot \mathcal{L}\{\cos 2t\}$

$= \boxed{\frac{2}{s^3} \cdot \frac{s}{s^2+4}}$

(d) $f(t) = t \cos 3t$

$\mathcal{L}\{t \cos 3t\} = (-1) \frac{d}{ds} \{\mathcal{L}\{\cos 3t\}\}$

$= -\frac{d}{ds}\left(\frac{s}{s^2+9}\right)$

$= -\frac{s^2+9-2s(s)}{(s^2+9)^2}$

$= \boxed{\frac{s^2-9}{(s^2+9)^2}}$

(e) $f(t) = e^t \delta(t-1)$

$\mathcal{L}\{\delta(t-1)\} = e^{-s}$

$\mathcal{L}\{e^t \delta(t-1)\} = \boxed{e^{-(s-1)}}$

7. Find the inverse Laplace transform of the given function.

(a) $F(s) = \frac{2s+6}{s^2-4s+8}$

$$\frac{2s+6}{s^2-4s+8} = \frac{2s+6}{(s-2)^2+4} = 2\,\frac{s+3}{(s-2)^2+4}$$

$$= 2\,\frac{s-2+5}{(s-2)^2+4} = 2\,\frac{s-2}{(s-2)^2+4} + 5\cdot\frac{2}{(s-2)^2+4}$$

$$\mathcal{L}^{-1}\{F(s)\} = 2\,\mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+4}\right\} + 5\,\mathcal{L}^{-1}\left\{\frac{2}{(s-2)^2+4}\right\}$$

$$= \boxed{2e^{2t}\cos 2t + 5e^{2t}\sin 2t}$$

(b) $F(s) = \frac{e^{-2s}}{s^2+s-2}$

$$\frac{1}{s^2+s-2} = \frac{1}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1}$$

$$= \frac{A(s-1)+B(s+2)}{(s+2)(s-1)}$$

$$1 = A(s-1)+B(s+2)$$

$s=1$: $1 = 3B \Rightarrow B = 1/3$

$s=-2$: $1 = -3A \Rightarrow A = -1/3$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+s-2}\right\} = \mathcal{L}^{-1}\left\{-\frac{1}{3}\,\frac{1}{s+2} + \frac{1}{3}\,\frac{1}{s-1}\right\}$$

$$= -\frac{1}{3}e^{-2t} + \frac{1}{3}e^{t}$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2+s-2}\right\} = \boxed{\left[-\frac{1}{3}e^{-2(t-2)} + \frac{1}{3}e^{(t-2)}\right]u_2(t)}$$

3. Solve the initial value problem using the Laplace transform:

(a) $y'' + 4y = \begin{cases} t, & 0 \le t < 1 \\ 1, & t \ge 1 \end{cases}$, $y(0) = y'(0) = 0$

$$g(t) = t + (1-t)u_1(t)$$

$$\mathcal{L}\{g(t)\} = \frac{1}{s^2} - \mathcal{L}\{(t-1)u_1(t)\}$$

$$= \frac{1}{s^2} - e^{-s}\mathcal{L}\{t\}$$

$$= \frac{1}{s^2} - e^{-s}\frac{1}{s^2}$$

$$\mathcal{L}\{y(t)\} = Y(s)$$

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY(s)$$

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s)$$

$$\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{g(t)\}$$

$$(s^2+4)Y(s) = \frac{1}{s^2} - \frac{e^{-s}}{s^2}$$

$$Y(s) = \frac{1}{s^2(s^2+4)} - \frac{e^{-s}}{s^2(s^2+4)}$$

Partial fractions:

$$\frac{1}{s^2(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4}$$

$$= \frac{As(s^2+4) + B(s^2+4) + (Cs+D)s^2}{s^2(s^2+4)}$$

$$1 = s^3(A+C) + s^2(B+D) + s(4A) + 4B$$

$s^3$: $0 = A + C$

$s^2$: $0 = B + D$ $\Longrightarrow$ $A = C = 0$, $B = 1/4$, $D = -1/4$

$s$: $0 = 4A$

$1$: $1 = 4B$

$$\frac{1}{s^2(s^2+4)} = \frac{1}{4}\frac{1}{s^2} - \frac{1}{4}\frac{1}{s^2+4}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\} = \frac{1}{4}t - \frac{1}{8}\sin 2t$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2(s^2+4)}\right\} = u_1(t)\left[\frac{1}{4}(t-1) - \frac{1}{8}\sin 2(t-1)\right]$$

$$y(t) = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+4)} - \frac{e^{-s}}{s^2(s^2+4)}\right\}$$

$$= \boxed{\frac{1}{4}t - \frac{1}{8}\sin 2t + u_1(t)\left[\frac{1}{4}(t-1) - \frac{1}{8}\sin 2(t-1)\right]}$$

(b) $y'' + 2y' + 3y = \delta(t - 3\pi)$, $y(0) = y'(0) = 0$

$$\mathcal{L}\{y'' + 2y' + 3y\} = \mathcal{L}\{\delta(t-3\pi)\}$$

$$\mathcal{L}\{y\} = Y(s)$$

$$\mathcal{L}\{y'\} = sY(s) - y(0)$$
$$= sY(s)$$

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0)$$
$$= s^2 Y(s)$$

$$\mathcal{L}\{\delta(t-3\pi)\} = e^{-3\pi s}$$

$$(s^2+2s+3)\,Y(s) = e^{-3\pi s}$$

$$Y(s) = \frac{1}{s^2+2s+3}\, e^{-3\pi s}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{e^{-3\pi s}}{s^2+2s+3}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+3}\right\}$$
$$= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+2}\right\}$$
$$= \frac{1}{\sqrt{2}}\,\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{(s+1)^2+2}\right\}$$
$$= \frac{1}{\sqrt{2}}\, e^{-t} \sin\sqrt{2}\,t$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-3\pi s}}{s^2+2s+3}\right\}$$
$$= \boxed{u_{3\pi}(t)\,\frac{1}{\sqrt{2}}\, e^{-(t-3\pi)} \sin\sqrt{2}(t-3\pi) = y(t)}$$

(c) $y'' + 4y' + 4y = g(t)$, $y(0) = 2$, $y'(0) = -3$

$\mathcal{L}\{y'' + 4y' + 4y\} = \mathcal{L}\{g(t)\}$

$\mathcal{L}\{g(t)\} = G(s)$

$\mathcal{L}\{y\} = Y(s)$

$\mathcal{L}\{y'\} = sY(s) - y(0) = sY(s) - 2$

$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0)$

$= s^2Y(s) - 2s + 3$

$s^2Y(s) - 2s + 3 + 4sY(s) - 8 + 4Y(s) = G(s)$

$Y(s)(s^2+4s+4) = G(s) + 2s + 5$

$Y(s) = \frac{G(s)}{s^2+4s+4} + \frac{2s+5}{s^2+4s+4}$

$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{G(s)}{s^2+4s+4} + \frac{2s+5}{s^2+4s+4}\right\}$

$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\}$

$= e^{-2t}t$

$\frac{2s+5}{s^2+4s+4} = \frac{A}{s+2} + \frac{B}{(s+2)^2}$

$= \frac{A(s+2) + B}{(s+2)^2}$

$2s+5 = A(s+2) + B$

$s=-2: \quad 1 = B$

$s=0: \quad 5 = 2A + B, \; 2A = 5 - B = 4$

$A = 2$

$\frac{2s+5}{s^2+4s+4} = \frac{2}{s+2} + \frac{1}{(s+2)^2}$

$\mathcal{L}^{-1}\left\{\frac{2s+5}{s^2+4s+4}\right\} = 2e^{-2t} + e^{-2t}t$

$$\boxed{y(t) = \int_0^t g(t-\tau)e^{-2\tau}\tau\,d\tau + 2e^{-2t} + e^{-2t}t}$$

9. Transform the given equation into a system of first order equation, then in matrix notation.

(a) $e^t y'' + t^2 y' - \sin ty = 3\arctan t, \quad y(0) = 5, \quad y'(0) = 3.$

(b) $y^{(4)} - \cos ty = 0.$

(a) $e^t y'' + t^2 y' - \sin t y = 3\arctan t$

$y'' + t^2 e^{-t} y' - \sin t e^{-t} y = 3e^{-t} \arctan t$

$x_1 = y(t) \qquad x_1' = x_2$

$x_2 = y'(t)$

$x_2' = y'' = 3e^{-t} \arctan t - t^2 e^{-t} y' - \sin t e^{-t} y$

$= 3e^{-t} \arctan t - t^2 e^{-t} x_2 - \sin t e^{-t} x_1$

$$\boxed{\begin{cases} x_1' = x_2 \\ x_2' = 3e^{-t} \arctan t - t^2 e^{-t} x_2 - e^{-t} \sin t\, x_1 \end{cases}, \quad x_1(0) = 5,\ x_2(0) = 3}$$

matrix notation:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -e^{-t} \sin t & -t^2 e^{-t} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 3e^{-t} \arctan t \end{pmatrix}$$

(b) $y^{(4)} - \cos t\, y = 0$

$y = x_1$

$y' = x_2$

$y'' = x_3$

$y''' = x_4$

$$\boxed{\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = x_4 \\ x_4' = (\cos t)\, x_1 \end{cases}}$$

$x_4' = y^{(4)} = \cos t y = \cos t \cdot x_1$

matrix form:

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \cos t & 0 & 0 & 0 \end{pmatrix}$$

**10.** Find $A^{-1}$ if $A = \begin{pmatrix} 1+i & -1+2i \\ 3+2i & 2-i \end{pmatrix}$

$$\det A = (1+i)(2-i) - (-1+2i)(3+2i) = 2 - i + 2i - \overbrace{i^2}^{-1} + 3 + 2i - 6i - 4\overbrace{i^2}^{-1}$$

$$= 5 + 5 - 3i = 10 - 3i$$

$$\frac{1}{\det A} = \frac{1}{10-3i} = \frac{10+3i}{(10-3i)(10+3i)} = \frac{10+3i}{100-9i^2} = \frac{10+3i}{109}$$

$$\boxed{A^{-1} = \frac{10+3i}{109}\begin{pmatrix} 2-i & 1-2i \\ -3-2i & 1+i \end{pmatrix}}$$

5. Find $BA$ if $A = \begin{pmatrix} 1+i & -1+2i \\ 3+2i & 2-i \end{pmatrix}$, $B = \begin{pmatrix} i & 3 \\ 2 & -2i \end{pmatrix}$

$$BA = \begin{pmatrix} i & 3 \\ 2 & -2i \end{pmatrix}\begin{pmatrix} 1+i & -1+2i \\ 3+2i & 2-i \end{pmatrix} = \begin{pmatrix} i(1+i)+3(3+2i) & i(-1+2i)+3(2-i) \\ 2(1+i)-2i(3+2i) & 2(-1+2i)-2i(2-i) \end{pmatrix}$$

$$= \begin{pmatrix} i+i^2+9+6i & -i+2i^2+6-3i \\ 2+2i-6i-4i^2 & -2+4i-4i+4i^2 \end{pmatrix} = \boxed{\begin{pmatrix} 8+7i & 4-4i \\ 6-4i & -6 \end{pmatrix}}$$

9. Find all eigenvalues and eigenvectors of the martix

$$\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

$$A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

$$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix}$$

$$= (-\lambda)(3-\lambda)^2 + 16 + 16 + 16\lambda - 4(3-\lambda) - 4(3-\lambda)$$

$$= -\lambda^3 + 6\lambda^2 + 15\lambda + 8 = 0$$

$$\lambda^3 - 6\lambda^2 - 15\lambda - 8 = 0.$$

plug $\lambda = -1$: $-1 - 6 + 15 - 8 = 0$

one eigenvalue is $\lambda_1 = -1$.

divide by long divisions:

$$\begin{array}{r|l} & \lambda^2 - 7\lambda + 8 \\ \lambda + 1 & \lambda^3 - 6\lambda^2 - 15\lambda - 8 \\ & \underline{\lambda^3 + \lambda^2} \\ & -7\lambda^2 - 15\lambda \\ & \underline{-7\lambda^2 - 7\lambda} \\ & -8\lambda - 8 \\ & \underline{-8\lambda - 8} \\ & 0. \end{array}$$

$$\lambda^3 - 6\lambda^2 - 15\lambda - 8 = (\lambda+1)(\lambda^2 - 7\lambda + 8) = (\lambda+1)^2(\lambda-8)$$

Eigenvalues : $\lambda_1 = -1$ — repeated

$\lambda_2 = 8$

corresponding eigenvectors:

$\lambda_1 = -1$

$\vec{V} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ is a solution to

$(A+I)\vec{V} = \vec{0}$

$$\begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 4v_1 + 2v_2 + 4v_3 = 0 \\ 2v_1 + v_2 + 2v_3 = 0 \\ 4v_1 + 2v_2 + 4v_3 = 0 \end{cases} \Rightarrow v_2 = 2v_3 + 2v_1$$

$$\vec{V} = \begin{pmatrix} v_1 \\ 2v_1 + 2v_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ 2v_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2v_3 \\ v_3 \end{pmatrix}$$

$$= v_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + v_3 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.$$

if we plug $v_1 = 0$ and $v_3 = 1$, then we'll get $\cancel{\vec{v}_1 =}$ the vector $\vec{x}_1 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

if we plug $v_1 = 1$ and $v_3 = 1$, then we'll get the vector $\vec{x}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$.

the eigenvalue $\lambda_1 = 1$ has <u>two</u> corresponding eigenvectors: $\vec{x}_1 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ and $\vec{x}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

$\lambda_2 = 8.$

$\vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ is a solution of

$(A - 8I)\vec{w} = \vec{0}$

$$\begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -5w_1 + 2w_2 + 4w_3 = 0 \\ w_1 - 4w_2 + w_3 = 0 \\ 4w_1 + 2w_2 - 5w_3 = 0 \end{cases}$$

Solve the 2nd equation for $w_3$

$$w_3 = 4w_2 - w_1$$

and then plug into the 1st and the 3rd

$$\begin{cases} -9w_1 + 18w_2 = 0 \\ 9w_1 - 18w_2 = 0 \end{cases} \Rightarrow \quad w_1 = 2w_2$$

and $\quad w_3 = 4w_2 - 2w_2 = 2w_2.$

$$\vec{w} = \begin{pmatrix} 2w_2 \\ w_2 \\ 2w_2 \end{pmatrix} = w_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$\boxed{\vec{x}_3 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \text{ eigenvector corresponding to } \lambda_2 = 8}$