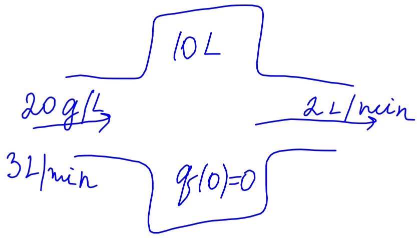
MATH 308-518,519,520 Spring 2013 Practice Test I

over sections 1.1–1.3, 2.1–2.6, 3.1

1. A large tank initially contains 10 L of fresh water. A brine containing 20 g/L of salt flows into the tank at a rate of 3 L/min. The solution inside the tank is kept well stirred and flows out of the tank at the rate 2 L/min. Determine the concentration of salt in the tank as a function of time.

$q(t)$ is the amount of salt in the tank at time t



$$\frac{dq}{dt} = \text{rate in} - \text{rate out}$$

$$\frac{dq}{dt} = 3(20) - \frac{2q(t)}{10+(3-2)t}$$

IVP: $$\begin{cases} \dfrac{dq}{dt} = 60 - \dfrac{2}{10+t}q \\ q(0)=0 \end{cases}$$ linear not separable

$$\frac{dq}{dt} + \frac{2}{10+t}q = 60$$

Integrating factor: $\dfrac{d\mu}{dt} = \dfrac{2}{10+t}\mu$

$$\int \frac{d\mu}{\mu} = \int \frac{2dt}{10+t} \qquad \ln|\mu| = 2\ln|10+t|$$

$$\mu(t) = (10+t)^2$$

$$(10+t)^2 q(t) = \int (10+t)^2(60)\,dt$$

$$= 60\,\frac{(10+t)^3}{3} + C$$

$$q(t) = 20(10+t) + \frac{C}{(10+t)^2}$$

$$0 = q(0) = 200 + \frac{C}{100}$$

$$C = -20000$$

$$q(t) = 20(10+t) - \frac{20000}{(10+t)^2}$$

$$\text{Concentration} = \frac{q(t)}{10+t} = \boxed{20 - \frac{20000}{(10+t)^3}}$$

2. Suppose that a sum S_0 is invested at an annual rate of return r compounded continuously.

(a) Find the time T required for the original sum to double in value as a function of r.

$$\frac{dS}{dt} = rS$$

$$S(0) = S_0$$

T such that

$$S(T) = 2S_0$$

$$S(t) = S_0 e^{rt}$$

$$T = \frac{\ln 2}{r}$$

(b) Determine T if $r = 7\%$. $= 0.07$

$$T = \frac{\ln 2}{0.07}$$

(c) Find the return rate that must be achieved if the initial investment is to double in 8 years.

$$T = \frac{\ln 2}{r}$$

$$8 = \frac{\ln 2}{r}$$

$$r = \frac{\ln 2}{8}$$

3. An object with temperature 150^0 is placed in a freezer whose temperature is 30^0. Assume that the temperature of the freezer remains essentially constant.

(a) If the object is cooled to 120^0 after 8 min, what will its temperature be after 18 min?

$T(t)$ is the temperature of the object at time t.

$T(0) = 150$

$M(t) = 30$ (outside temperature)

$$\frac{dT}{dt} = k(30 - T(t))$$

$$\int \frac{dT}{30 - T(t)} = \int k\,dt$$

$$-\ln|30 - T| = kt + C$$

$$\ln|30 - T| = -kt + C_1$$

$$30 - T = C_2 e^{-kt}$$

$$T(t) = 30 - C_2 e^{-kt}$$

$$T(0) = 150 = 30 - C_2$$

$$C_2 = -120$$

$$T(t) = 30 + 120 e^{-kt}$$

$$T(8) = 120$$

$$120 = T(8) = 30 + 120 e^{-8k}$$

$$e^{-8k} = \frac{3}{4}$$

$$k = -\frac{1}{8} \ln \frac{3}{4} \approx 0.036$$

$$\boxed{T(t) = 30 + 120 e^{-0.036 t}}$$

(b) When will its temperature be 60^0?

Find t_0 such that

$$60 = T(t_0) = 30 + 120e^{-0.036t}$$

$$e^{-0.036t} = \frac{1}{4}$$

$$\boxed{t = \frac{-1}{0.036} \ln\left(\frac{1}{4}\right)}$$

4. Determine (without solving the problem) an interval in which the solution to the initial value problem

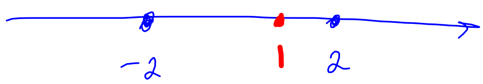
$$(4-t^2)y' + 2ty = 3t^2, \quad y(1) = -3$$

is certain to exist.

$$y' + \frac{2t}{4-t^2}\, y = \frac{3t^2}{4-t^2}$$

continuous if $4-t^2 \neq 0$

or $t \neq \pm 2$



$$\boxed{(-2,2)}$$

Find the point (if any) at which the direction field of the equation

$$y' = (y-1)(y+2)(t^2-2t+1)$$

is horizontal.

$$(y-1)(y+2)(t^2-2t+1) = 0.$$

$$y=1, \quad y=-2, \quad (t-1)^2 = 0.$$

$$y=1, \quad y=-2, \quad t=1$$

$(1,1)$

$(1,-2)$

$\left.\begin{array}{l}(t, 1) \\ (t,-2)\end{array}\right\}$ any t

$(1, y)$ - any y

5. Solve the initial value problem

$$y' = \frac{t^2}{1+t^3}, \qquad y(0) = y_0$$

and determine how the interval in which the solution exists depends on the initial value y_0.

#5a. $\quad y' = -\dfrac{4t}{y}, \quad y(0) = y_0$

$$y \neq 0$$

IVP: $\quad \dfrac{dy}{dt} = -\dfrac{4t}{y}, \quad y(0) = y_0$

$$\int y\,dy = -\int 4t\,dt$$

$$\frac{y^2}{2} = -2t^2 + C$$

$$y^2 = -4t^2 + C_1$$

$y > 0$	$y < 0$
$y(t) = \sqrt{-4t^2 + C_1}$	$y(t) = -\sqrt{-4t^2 + C_1}$
$y(0) = y_0$	

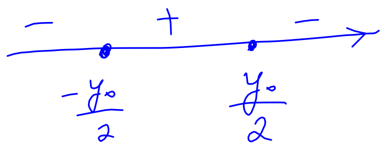
$$y_0 = \pm\sqrt{C_1}$$

$$C_1 = y_0^2$$

$$y(t) = \pm\sqrt{-4t^2 + y_0^2}$$

$$-4t^2 + y_0^2 \geq 0$$

$$(y_0 - 2t)(y_0 + 2t) \geq 0$$



$$\boxed{-\frac{y_0}{2} \leq t \leq \frac{y_0}{2}}$$

6. Solve the following initial value problem

$$\sqrt{y}dt + (1+t)dy = 0 \quad y(0) = 1.$$

separable equation.

$$(1+t)\,dy = -\sqrt{y}\,dt$$

$$\int \frac{dy}{\sqrt{y}} = -\int \frac{dt}{1+t}$$

$$2y^{1/2} = -\ln|1+t| + C$$

$$y^{1/2} = C_1 - \frac{1}{2}\ln|1+t|$$

$$y = \left[C_1 - \frac{1}{2}\ln|1+t|\right]^2$$

$$1 = y(0) = C_1^2, \quad C_1 = \pm 1$$

$$\boxed{y(t) = \left[\pm 1 - \frac{1}{2}\ln|1+t|\right]^2}$$

7. Find the general solution to the equation

$$(t^2 - 1)y' + 2ty + 3 = 0$$

$$y' + \frac{2t}{t^2-1}\,y = -\frac{3}{t^2-1}$$

Integrating factor:

$$\frac{d\mu}{dt} = \frac{2t}{t^2-1}\,\mu$$

$$\int \frac{d\mu}{\mu} = \int \frac{2t\,dt}{t^2-1}$$

$$\ln|\mu| = \ln|t^2-1|$$

$$\mu(t) = t^2-1$$

$$(t^2-1)\,y(t) = \int \cancel{(t^2-1)}\left(-\frac{3}{\cancel{t^2-1}}\right)dt$$

$$= -3t + C$$

$$\boxed{y(t) = -\frac{3t}{t^2-1} + \frac{C}{t^2-1}}$$

8. Solve the initial value problem

$$\underbrace{(ye^{xy}\cos(2x) - 2e^{xy}\sin(2x) + 2x)}_{M(x,y)}dx + \underbrace{(xe^{xy}\cos(2x) - 3)}_{N(x,y)}dy = 0, \quad y(0) = -1$$

$M_y = N_x$ Exact.

$F(x,y)$:

$$\begin{cases} \frac{\partial F}{\partial x} = ye^{xy}\cos(2x) - 2e^{xy}\sin(2x) + 2x \\ \int \frac{\partial F}{\partial y}dy = \int (xe^{xy}\cos(2x) - 3)\,dy \end{cases}$$

$$F(x,y) = x\frac{1}{x}e^{xy}\cos(2x) - 3y + h(x)$$

$$= e^{xy}\cos(2x) - 3y + h(x)$$

$$\frac{\partial F}{\partial x} = ye^{xy}\cos(2x) - 2e^{xy}\sin(2x) + h'(x)$$

$$h'(x) = 2x$$

$$h(x) = x^2 + C$$

Update $F(x,y) = e^{xy}\cos(2x) - 3y + x^2 + C$

General solution: $\boxed{e^{xy}\cos(2x) - 3y + x^2 + C = 0}$

Plug $x=0$ and $y=-1$ into the general solution and then solve for C:

$$1 - 3(-1) + C = 0$$

$$C = -4$$

$$\boxed{e^{xy}\cos(2x) - 3y + x^2 - 4 = 0}$$

9. Find an integrating factor for the equation

$$(3xy + y^2) + (x^2 + xy)y' = 0$$

and then solve the equation. $y=0$ is a solution

$$\underbrace{(x^2+xy)}_{N(x,y)}\,dy + \underbrace{(3xy+y^2)}_{M(x,y)}\,dx = 0$$

$$\frac{M_y - N_x}{N(x,y)} = \frac{3x+2y-(2x+y)}{x^2+xy}$$

$$= \frac{x+y}{x(x+y)}$$

$$= \frac{1}{x} \quad \text{(depends on } x \text{ only)}$$

Integrating factor $\mu(x)$:

$$\frac{d\mu}{dx} = \frac{\mu}{x}$$

$$\int \frac{d\mu}{\mu} = \int \frac{dx}{x}$$

$$\mu(x) = x$$

$$x\left[(3xy+y^2)dx + (x^2+xy)dy = 0 \right]$$

$$(3x^2y + y^2x)dx + (x^3+x^2y)\,dy = 0 \quad \text{Exact}$$

$$\begin{cases} \int \frac{\partial F}{\partial x} dx = \int 3x^2y + y^2x\,dx \\ \frac{\partial F}{\partial y} = x^3 + x^2y \end{cases}$$

$$F(x,y) = x^3y + \frac{x^2y^2}{2} + h(y)$$

$$\frac{\partial F}{\partial y} = x^3 + \frac{2x^2y}{2} + h'(y)$$

$$h'(y) = 0 \quad \text{or} \quad h(y) = C$$

$$F(x,y) = x^3y + \frac{x^2y^2}{2} + C$$

General solution: $\boxed{x^3y + \frac{x^2y^2}{2} + C = 0}$

10. Solve the initial value problem

$$6y'' - 5y' + y = 0, \qquad y(0) = 4, y'(0) = 0$$

auxiliary equation: $6r^2 - 5r + 1 = 0$

$y'' \to r^2$
$y' \to r$
$y \to 1$

$$r_1 = \frac{5 + \sqrt{25 - 24}}{12} = \frac{1}{2}$$

$$r_2 = \frac{5 - 1}{12} = \frac{1}{3}$$

General solution:

$$y(t) = c_1 e^{t/2} + c_2 e^{t/3}$$

$$y(0) = c_1 + c_2 = 4$$

$$y'(t) = \frac{c_1}{2} e^{t/2} + \frac{c_2}{3} e^{t/3}$$

$$y'(0) = \frac{c_1}{2} + \frac{c_2}{3} = 0$$

$$c_1 + c_2 = 4$$

$$c_1 = -\frac{2}{3} c_2$$

$$c_2 = 12$$

$$c_1 = -8$$

$$\boxed{y(t) = -8e^{t/2} + 12e^{t/3}}$$

11. Find the general solution to the equation

$$4y'' - 12y' + 9y = 0$$

auxiliary equation:

$$4r^2 - 12r + 9 = 0$$

$$(2r-3)^2 = 0$$

$r = \frac{3}{2}$ repeated root

General solution:

$$\boxed{y(t) = (c_1 + c_2 t)e^{3t/2}}$$

11. Find the general solution to the equation

$$4y'' - 12y' + 9y = 0$$

auxiliary equation:

$$4r^2 - 12r + 9 = 0$$

$$(2r-3)^2 = 0$$

$r = \frac{3}{2}$ repeated root

General solution:

$$\boxed{y(t) = (c_1 + c_2 t)e^{3t/2}}$$

. Consider the initial value problem

$$y' + 2y = 5 - t, \qquad y(0) = y_0$$

Find the value y_0 for which the solution touches, but does not cross the t-axis.

$$\frac{d\mu}{dt} = 2\mu$$

$$\frac{d\mu}{\mu} = 2\,dt$$

$$\mu(t) = e^{2t}$$

$$e^{2t} y(t) = \int \underbrace{(5-t)}_{u} \underbrace{e^{2t}dt}_{dv}$$

Integrate by parts:

$du = -dt \qquad v = \frac{1}{2}e^{2t}$

$$e^{2t} y(t) = \frac{5-t}{2} e^{2t} - \int \frac{1}{2} e^{2t} (-dt)$$

$$= \frac{5-t}{2} e^{2t} + \frac{1}{4} e^{2t} + C$$

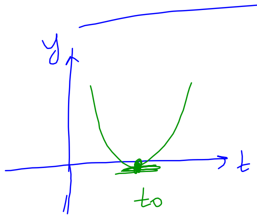
$$y(t) = \frac{5-t}{2} + \frac{1}{4} + Ce^{-2t}$$

$$= \frac{11}{4} - \frac{t}{2} + Ce^{-2t}$$

$$y(0) = \frac{11}{4} + C = y_0$$

$$C = y_0 - \frac{11}{4}$$

$$y(t) = \frac{11}{4} - \frac{t}{2} + \left(y_0 - \frac{11}{4}\right)e^{-2t}$$



Point at which $y(t)$ touches but does not intersect the t-axis is: slope of the direction field is 0

$$y(t_0) = 0$$

$$y'(t_0) = 0$$

$$y'(t_0) + 2y(t_0) = 5 - t_0$$

$$\underbrace{y'(t_0)}_{0} + 2\underbrace{y(t_0)}_{0} = 5 - t_0$$

$$t_0 = 5$$

$$y(5) = 0$$

$$0 = y(5) = \frac{11}{4} - \frac{5}{2} + \left(y_0 - \frac{11}{4}\right)e^{-10}$$

$$\boxed{y_0 = \frac{11 - e^{10}}{4}}$$