1. Find the general solution of the system.

(a) $\mathbf{x}'(t) = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}\mathbf{x}$, $A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$, $A - \lambda I = \begin{pmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{pmatrix}$

Eigenvalues: $\det(A - \lambda I) = 0$

$(3-\lambda)(-1-\lambda) + 8 = 0$

$-3 - 3\lambda + \lambda + \lambda^2 + 8 = 0$

$\lambda^2 - 2\lambda + 5 = 0$

$\lambda_1 = \dfrac{2 + \sqrt{4-20}}{2}$

$= \dfrac{2 + \sqrt{-16}}{2}$

$= 1 + 2i$, $\lambda_2 = 1 - 2i$

Eigenvector: $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ corresponds to $\lambda_1 = 1 + 2i$

$\vec{v}$ is a solution of the system

$(A - \lambda_1 I)\vec{v} = \vec{0}$

$\begin{pmatrix} 3-(1+2i) & -2 \\ 4 & -1-(1+2i) \end{pmatrix}\vec{v} = \vec{0}$

$\begin{pmatrix} 2-2i & -2 \\ 4 & -2-2i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{cases} (2-2i)v_1 - 2v_2 = 0 \\ 4v_1 - (2+2i)v_2 = 0 \end{cases}$ $(1-i)v_1 - v_2 = 0$, $v_2 = (1-i)v_1$

$\vec{v} = \begin{pmatrix} v_1 \\ (1-i)v_1 \end{pmatrix} \overset{v_1 = 1}{=} \begin{pmatrix} 1 \\ 1-i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \end{pmatrix}$

Solution: $\vec{v}e^{\lambda_1 t} = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right]e^{(1+2i)t}$ $\boxed{e^{(1+2i)t} = e^t(\cos 2t + i\sin 2t)}$

$= \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right]e^t(\cos 2t + i\sin 2t)$

$= e^t\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos 2t + i\begin{pmatrix} 1 \\ 1 \end{pmatrix}\sin 2t + i\begin{pmatrix} 0 \\ -1 \end{pmatrix}\cos 2t + i^2\begin{pmatrix} 0 \\ -1 \end{pmatrix}\sin 2t\right]$

$= e^t\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos 2t - \begin{pmatrix} 0 \\ -1 \end{pmatrix}\sin 2t + i\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\sin 2t + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\cos 2t\right]\right]$

$\vec{x}_1(t) = Re(\vec{v}e^{\lambda_1 t}) = e^t\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos 2t - \begin{pmatrix} 0 \\ -1 \end{pmatrix}\sin 2t\right] = e^t\begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix}$

$\vec{x}_2(t) = Im(\vec{v}e^{\lambda_1 t}) = e^t\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\sin 2t + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\cos 2t\right] = e^t\begin{pmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{pmatrix}$

General solution:

$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) = \boxed{e^t\left[c_1\begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + c_2\begin{pmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{pmatrix}\right]}$

(b) $\mathbf{x}'(t) = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \mathbf{x}$. , $A = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$, $A - \lambda I = \begin{pmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{pmatrix}$

$$\det(A-\lambda I) = \begin{vmatrix} (1-\lambda) & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)(-1-\lambda) + 2 + 12 - 8(2-\lambda) - 3(1+\lambda) + 1(1-\lambda)$$

$$= -(1-\lambda)(2-\lambda)(1+\lambda) + 14 - 16 + 8\lambda - 3 - 3\lambda + 1 - \lambda$$

$$= -(2-\lambda)(1-\lambda^2) + 4\lambda - 4$$

$$= -2 + 2\lambda^2 + \lambda - \lambda^3 + 4\lambda - 4$$

$$= -\lambda^3 + 2\lambda^2 + 5\lambda - 6 = 0$$

$$\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$$

$\lambda = 1$: $\quad 1 - 2 - 5 + 6 = 0$

$$\begin{array}{r} \lambda^2 - \lambda - 6 \\ \lambda - 1 \,\big)\, \overline{\lambda^3 - 2\lambda^2 - 5\lambda + 6} \\ \underline{\lambda^3 - \lambda^2 \qquad\qquad} \\ -\lambda^2 - 5\lambda \quad \\ \underline{-\lambda^2 + \lambda \quad} \\ -6\lambda + 6 \end{array}$$

$$(\lambda - 1)(\lambda^2 - \lambda - 6) = 0$$

$$(\lambda - 1)(\lambda - 3)(\lambda + 2) = 0$$

$\lambda_1 = 1, \lambda_2 = 3, \lambda_3 = -2$

corresponding eigenvectors.

1) $\lambda_1 = 1$. $\quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$(A - \lambda_1 I)\vec{v} = \vec{0}$

$$\begin{pmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -v_2 + 4v_3 = 0 \\ 3v_1 + v_2 - v_3 = 0 \\ 2v_1 + v_2 - 2v_3 = 0 \end{cases} \qquad v_2 = 4v_3$$

$3v_1 + 4v_3 - v_3 = 0$

$v_1 = -v_3$

$2(-v_3) + 4v_3 - 2v_3 = 0$ for all v_3

$$\vec{v} = \begin{pmatrix} -v_3 \\ 4v_3 \\ v_3 \end{pmatrix} \overset{v_3=1}{=} \boxed{\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} \text{ corresponds to } \lambda_1 = 1}$$

2) $\lambda_2 = 3$ $\quad \vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$

$(A - \lambda_2 I)\vec{w} = \vec{0}$

$$\begin{pmatrix} -2 & -1 & 4 \\ 3 & -1 & -1 \\ 2 & 1 & -4 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -2w_1 - w_2 + 4w_3 = 0 \\ 3w_1 - w_2 - w_3 = 0 \\ 2w_1 + w_2 - 4w_3 = 0 \end{cases} \qquad w_2 = 4w_3 - 2w_1$$

$3w_1 - (4w_3 - 2w_1) - w_3 = 0$

$5w_1 - 5w_3 = 0$ $\quad \boxed{w_1 = w_3}$ $\boxed{w_2 = 2w_3}$

$$\vec{w} = \begin{pmatrix} w_3 \\ 2w_3 \\ w_3 \end{pmatrix} \overset{w_3=1}{=} \boxed{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \text{ corresponds to } \lambda_2 = 3}$$

$\lambda_3 = -2$. $\quad \vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$(A - \lambda_3 I)\vec{y} = \vec{0}$

$$\begin{pmatrix} 3 & -1 & 4 \\ 3 & 4 & -1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 3y_1 - y_2 + 4y_3 = 0 \\ 3y_1 + 4y_2 - y_3 = 0 \\ 2y_1 + y_2 + y_3 = 0 \end{cases} \qquad \begin{array}{l} \text{1st - 2nd} \\ -5y_2 + 5y_3 = 0 \\ \boxed{y_2 = y_3} \end{array}$$

$2y_1 + 2y_3 = 0$

$\boxed{y_1 = -y_3}$

$$\vec{y} = \begin{pmatrix} -y_3 \\ y_3 \\ y_3 \end{pmatrix} \overset{y_3=1}{=} \boxed{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \text{ corresponds to } \lambda = -2}$$

General solution: $\boxed{\vec{x}(t) = c_1 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} e^{3t} + c_3 \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} e^{t}}$

(c) $\mathbf{x}'(t) = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix} \mathbf{x}.$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix}, \quad A - \lambda I = \begin{pmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{pmatrix}$$

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 + 4(1-\lambda) = 0$$

$$(1-\lambda)\left[(1-\lambda)^2 + 4\right] = 0$$

$$\lambda_1 = 1 \qquad (1-\lambda)^2 + 4 = 0$$

$$(1-\lambda)^2 = -4$$

$$1-\lambda = 2i \qquad 1-\lambda = -2i$$

$$\lambda_2 = 1-2i \qquad \lambda_3 = 1+2i$$

Eigenvectors:
$\lambda_1 = 1$

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$(A-I)\vec{v} = \vec{0}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 2v_1 - 2v_3 = 0 \\ 3v_1 + 2v_2 = 0 \end{cases} \qquad v_1 = v_3, \quad v_2 = -\frac{3}{2} v_1$$

$$\vec{v} = \begin{pmatrix} v_1 \\ -3/2\, v_1 \\ v_1 \end{pmatrix} \underset{=}{\overset{v_1 = 1}{=}} \begin{pmatrix} 1 \\ -3/2 \\ 1 \end{pmatrix}$$

$$\lambda_3 = 1+2i \qquad \vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

$$\left(A - (1+2i)I\right)\vec{w} = \vec{0}$$

$$\begin{pmatrix} 1-(1+2i) & 0 & 0 \\ 2 & 1-(1+2i) & -2 \\ 3 & 2 & 1-(1+2i) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2i\, w_1 = 0 \qquad \boxed{w_1 = 0}$$

$$\begin{cases} 2w_1 - 2i\,w_2 - 2w_3 = 0 \\ 3w_1 + 2w_2 - 2i\,w_3 = 0 \end{cases} \qquad w_3 = -i\,w_2$$

$$\vec{w} = \begin{pmatrix} 0 \\ w_2 \\ -i\,w_2 \end{pmatrix} \overset{w_2 = 1}{=} \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\vec{w}\, e^{\lambda_3 t} = e^t \left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cos 2t - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \sin 2t + i \left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \sin 2t + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cos 2t \right] \right]$$

$$\mathrm{Re}\left(\vec{w}\, e^{\lambda_3 t}\right) = e^t \begin{pmatrix} 0 \\ \cos 2t \\ -\sin 2t \end{pmatrix}$$

$$\mathrm{Im}\left(\vec{w}\, e^{\lambda_3 t}\right) = e^t \begin{pmatrix} 0 \\ \sin 2t \\ \cos 2t \end{pmatrix}$$

General solution:

$$\boxed{\vec{x}(t) = C_1 e^t \begin{pmatrix} 0 \\ \cos 2t \\ -\sin 2t \end{pmatrix} + C_2 e^t \begin{pmatrix} 0 \\ \sin 2t \\ \cos 2t \end{pmatrix} + C_3 e^t \begin{pmatrix} 1 \\ -3/2 \\ 1 \end{pmatrix}}$$

(d) $\mathbf{x}'(t) = \begin{pmatrix} 5 & 2 & 4 \\ 2 & 2 & 2 \\ 4 & 2 & 5 \end{pmatrix} \mathbf{x}$.

$$A = \begin{pmatrix} 5 & 2 & 4 \\ 2 & 2 & 2 \\ 4 & 2 & 5 \end{pmatrix}, \quad A - \lambda I = \begin{pmatrix} 5-\lambda & 2 & 4 \\ 2 & 2-\lambda & 2 \\ 4 & 2 & 5-\lambda \end{pmatrix}$$

$$\det(A-\lambda I) = \begin{vmatrix} 5-\lambda & 2 & 4 \\ 2 & 2-\lambda & 2 \\ 4 & 2 & 5-\lambda \end{vmatrix} = (5-\lambda)^2(2-\lambda) + 16 + 16 - 16(2-\lambda) - 4(5-\lambda) - 4(5-\lambda)$$

$$= (25 - 10\lambda + \lambda^2)(2-\lambda) + 16\lambda - 40 + 8\lambda$$

$$= 50 - 20\lambda + 2\lambda^2 - 25\lambda + 10\lambda^2 - \lambda^3 + 24\lambda - 40$$

$$= -\lambda^3 + 12\lambda^2 - 21\lambda + 10 = 0$$

$$\lambda^3 - 12\lambda^2 + 21\lambda - 10 = 0$$

$$\begin{array}{r} \lambda^2 - 11\lambda + 10 \\ \lambda - 1 \,\big)\, \overline{\lambda^3 - 12\lambda^2 + 21\lambda - 10} \\ \underline{\lambda^3 - \lambda^2} \\ -11\lambda^2 + 21\lambda \\ \underline{-11\lambda^2 + 11\lambda} \\ 10\lambda - 10 \end{array}$$

$\lambda = 1$: $\quad 1 - 12 + 21 - 10 = 0$

$$(\lambda - 1)(\lambda^2 - 11\lambda + 10) = 0$$

$$(\lambda - 1)^2(\lambda - 10) = 0$$

$\lambda_1 = 10$ – nonrepeated

$\lambda_2 = 1$ – repeated

Eigenvectors: $\lambda_1 = 10$ $\quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$$(A - 10I)\vec{v} = \vec{0}$$

$$\begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -5v_1 + 2v_2 + 4v_3 = 0 \\ 2v_1 - 8v_2 + 2v_3 = 0 \\ 4v_1 + 2v_2 - 5v_3 = 0 \end{cases}$$

1st – 3rd

$-9v_1 + 9v_3 = 0$

$v_1 = v_3$

$$-8v_2 + 4v_1 = 0$$

$$v_2 = \frac{1}{2} v_1$$

$$\vec{v} = \begin{pmatrix} v_1 \\ \frac{1}{2}v_1 \\ v_1 \end{pmatrix} \overset{v_1 = 1}{=} \boxed{\begin{pmatrix} 1 \\ 1/2 \\ 1 \end{pmatrix} \text{ corresponds to } \lambda_1 = 10}$$

$\lambda_2 = 1$. $\quad \vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$

$$(A - I)\vec{w} = \vec{0}$$

$$\begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$2w_1 + w_2 + 2w_3 = 0$$

$$w_2 = -2w_1 - 2w_3$$

$$\vec{w} = \begin{pmatrix} w_1 \\ -2w_1 - 2w_3 \\ w_3 \end{pmatrix} = \begin{pmatrix} w_1 \\ -2w_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -2w_3 \\ w_3 \end{pmatrix}$$

$$= w_1 \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + w_3 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

General solution: $\boxed{\vec{x}(t) = C_1 \begin{pmatrix} 1 \\ 1/2 \\ 1 \end{pmatrix} e^{10t} + C_2 \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} e^{t} + C_3 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} e^{t}}$

2. Find the solution of the initial value problem.

(a) $\mathbf{x}'(t) = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}\mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$

$$\det(A-\lambda I) = \begin{vmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{vmatrix} = (5-\lambda)(1-\lambda)+3=0$$

$$5-6\lambda+\lambda^2+3=0$$

$$\lambda^2-6\lambda+8=0$$

$$(\lambda-2)(\lambda-4)=0$$

$$\lambda_1=2, \quad \lambda_2=4$$

Eigenvectors. $\lambda_1=2$, $\vec{v}=\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$$(A-2I)\vec{v}=\vec{0}$$

$$\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\boxed{3v_1=v_2}$$

$$\vec{v} = \begin{pmatrix} v_1 \\ 3v_1 \end{pmatrix} \overset{v_1=1}{=} \boxed{\begin{pmatrix} 1 \\ 3 \end{pmatrix} \text{ corresponds to } \lambda_1=2}$$

$\lambda_2=4$. $\vec{w}=\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

$$\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$w_1=w_2$$

$$\vec{w} = \begin{pmatrix} w_1 \\ w_1 \end{pmatrix} \overset{w_1=1}{=} \boxed{\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ corresponds to } \lambda_2=4}$$

general solution: $\boxed{\vec{x}(t) = C_1\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{2t} + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{4t}}$

$$\vec{x}(0) = C_1\begin{pmatrix} 1 \\ 3 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\begin{cases} C_1+C_2=2 \\ 3C_1+C_2=-1 \end{cases} \qquad C_2=2-C_1$$

$$3C_1+2-C_1=-1$$

$$2C_1=-3 \qquad \boxed{C_1=-3/2} \quad \boxed{C_2=7/2}$$

Solution of IVP:

$$\boxed{\vec{x}(t) = -\frac{3}{2}\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{2t} + \frac{7}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{4t}}$$

(b) $\mathbf{x}'(t) = \begin{pmatrix} 3 & -4 \\ 4 & -5 \end{pmatrix}\mathbf{x} \qquad \mathbf{x}(0) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$

(c) $\mathbf{x}'(t) = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$