Math 308 WEEK in REVIEW 1 Spring 2014

1. Given the following differential equations and their corresponding direction field, determine the behavior as $t \to \infty$

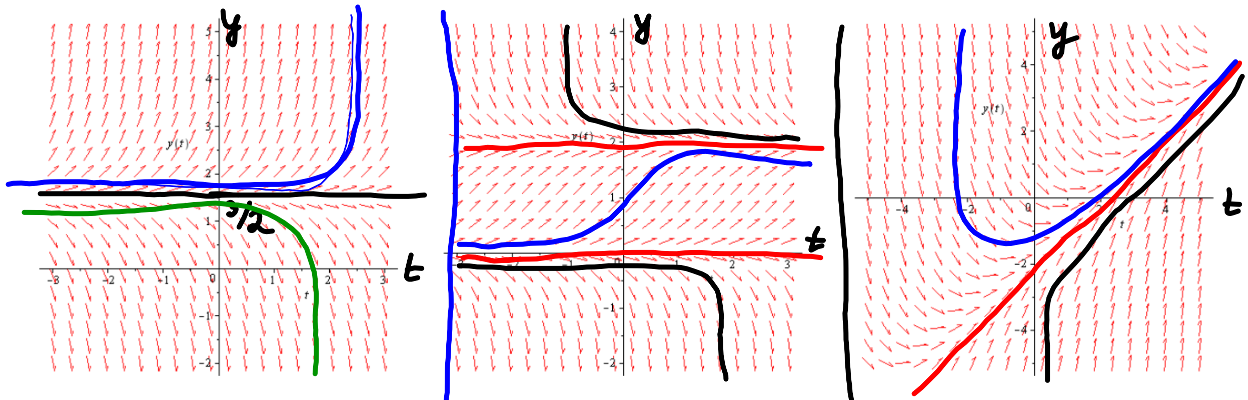


Fig. 1: $y'(t) = 2y(t) - 3$ Fig. 2: $y'(t) = y(t)(2 - y(t))$ Fig. 3: $y'(t) = t - 1 - y(t)$

$y(0) = y_0$ $y(0) = y_0$

Can we find the equation of the linear solution of the last differential equation?

$$\lim_{t\to\infty} y(t) = \begin{cases} \infty, & \text{if } y_0 > 3/2 \\ 3/2, & \text{if } y_0 = 3/2 \\ -\infty, & \text{if } y_0 < 3/2 \end{cases}$$

$$\lim_{t\to\infty} y(t) = \begin{cases} 2, & \text{if } y_0 > 0 \\ 0, & \text{if } y_0 = 0 \\ -\infty, & \text{if } y_0 < 0 \end{cases}$$

$$\lim_{t\to\infty} y(t) = \infty$$

$y' = t - 1 - y$

Find the linear solution of the equation.

General form of a linear solution is

$y = at + b$ $y' = a$

where a and b are unknown constants.

Plug y and y' into the equation:

$a = t - 1 - at - b$

$t - 1 - at - b - a = 0$

$t(1-a) - 1 - b - a = 0$

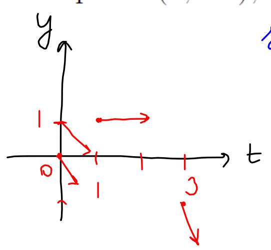
$$\left.\begin{aligned} t&: 1 - a = 0 \\ 1&: -1 - b - a = 0 \end{aligned}\right\} \quad \begin{aligned} &a = 1 \\ &a + b = -1 \\ &b = -1 - a = -2 \end{aligned}$$

$\boxed{y = t - 2}$ linear solution.

2. Given the differential equation

$$\frac{dy}{dt} = ty - 1.$$

(a) What is the slope of the graph of the solutions at $(0,1)$, at the point $(1,1)$, at the point $(3,-1)$, at the point $(0,0)$?



slope $= ty - 1$

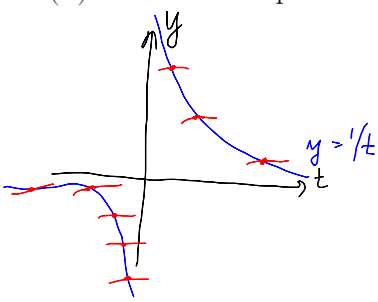
@ $(0,1)$ $= -1$

@ $(1,1)$ $= 0$

@ $(3,-1)$ $= -4$

@ $(0,0)$ $= -1$

(b) Find all the points where the tangents to the solution curves are horizontal? (slope = 0)



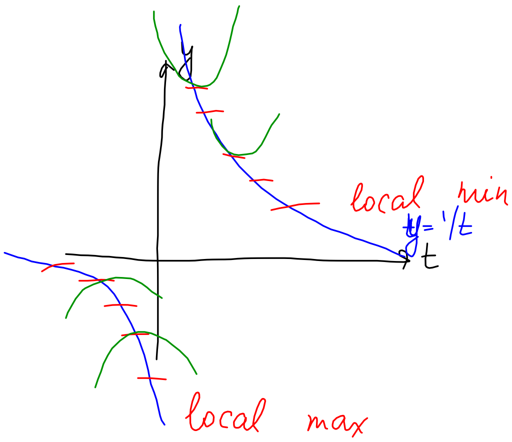
$ty - 1 = 0$

$y = \frac{1}{t}$

(c) Describe the nature of the critical points.

2nd derivative test: if $f'(t_0)=0$ and $f''(t_0)>0$ local min

$f''(t_0)<0$ local max



$$y' = ty - 1$$

$$\frac{d}{dt}(y') = \frac{d}{dt}(ty-1)$$

$$y'' = y - ty'$$

$$y''(t_0) = y(t_0) - t_0 \overbrace{y'(t_0)}^{0}$$

$$= y(t_0) = \begin{cases} >0, & \text{if } y>0 \\ <0, & \text{if } y<0 \end{cases}$$

3. The $\underbrace{\text{instantaneous rate of change of the temperature } T}_{= \frac{dT}{dt}}$ of coffee at time t is proportional to the difference between the temperature M of the air and the temperature T at time t.

(a) Find the mathematical model for the problem.

$$\frac{dT}{dt} = k(M-T), \quad k \text{ is a constant}$$

(b) Given that the room temperature is 75° and $k = 0.08$, find the solutions to the differential equation.

$$\frac{dT}{dt} = k(M-T)\quad, \qquad \text{(coffee is cooling down)}$$

$$\frac{dT}{dt} = 0.08(75-T), \quad T(t) \text{ is a function.}$$

separable equation:

$$\int \frac{dT}{75-T} = \int 0.08\,dt$$

$$-\ln|75-T| = 0.08t + C$$

$$\ln|75-T| = -0.08t - C$$

solve for $T(t)$:

$$75 - T(t) = e^{-0.08t - C}$$

$$T(t) = 75 - e^{-0.08t - C}$$

$$= 75 - e^{-0.08t} \cdot e^{-C} \quad = C_1$$

$$= \boxed{75 - C_1 e^{-0.08t}}$$

general solution.

(c) The initial temperature of the coffee is 200°F. Find the solution to the problem.

$$T(0) = 200$$

$$T(t) = 75 - C_1 e^{-0.08t}$$

$$200 = T(0) = 75 - C_1$$

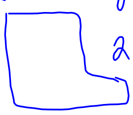
$$C_1 = -125$$

$\boxed{T(t) = 75 + 125 e^{-0.08t}}$ solution to IVP

4. Your swimming pool containing 60,000 gal of water has been contaminated by 5 kg of a non toxic dye that leaves a swimmer's skin an unattractive green. The pool's filtering system can take water from the pool, remove the dye, and return the water to the pool at a flow rate of 200 gal/min.

   (a) Write down the initial value problem for the filtering process; let $q(t)$ be the amount of dye in the pool at any time t.

Volume = 60,000 gal



200 gal/min

$q(0) = 5$

rate of change of $q(t)$ is $\frac{dq}{dt}$

$$\frac{dq}{dt} = -\frac{200}{60000} q(t)$$

concentration of the dye

initial value problem:

$$\frac{dq}{dt} = -\frac{1}{300} q(t)$$

$$q(0) = 5$$

(b) Solve the problem.

$$\frac{dq}{dt} = -\frac{1}{300} q, \quad q(0) = 5$$

separate variables

$$\frac{dq}{q} = -\frac{1}{300} dt$$

integrate

$$\int \frac{dq}{q} = -\frac{1}{300} \int dt$$

$$\ln|q| = -\frac{1}{300} t + c$$

$$q(t) = e^{-\frac{1}{300} t + c}$$

$$= \underbrace{e^{c}}_{=c_1} e^{-\frac{1}{300} t}$$

$$= c_1 e^{-\frac{t}{300}}$$

general solution

$$q(0) = c_1 = 5$$

$$\boxed{q(t) = 5e^{-\frac{1}{300} t}}$$

(c) You have invited several dozen friends to a pool party that is scheduled to begin in 4 hours. You have also determined that the effect of the dye us imperceptible if its concentration is less than 0.02 g/gal. Is your filtering system capable of reducing the dye concentration to this level within 4 hours? NO.

amount of dye in 4 hours (240 min)

$$q(240) = 5e^{-\frac{240}{300}} \approx 2.25$$

$$\text{concentration} = \frac{q(240)}{60000} = \frac{2.25}{60000} \approx 0.0000375 \ (kg/gal)$$

$$= 0.0375 \ (g/gal)$$

$$> 0.02$$

(d) Find the time T at which the concentration of dye first reaches the value 0.02 g/gal.

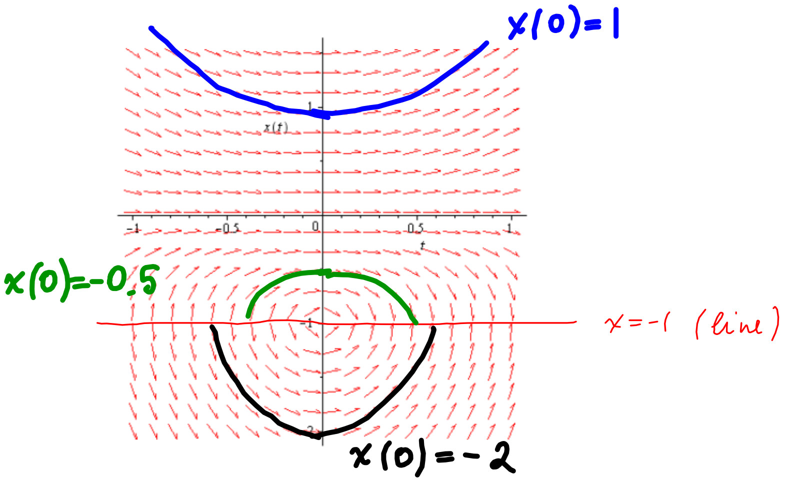
(e) Find the flow rate that is sufficient to achieve the concentration 0.02g/gal within 4 hours.

5. The direction field for the differential equation

$$x'(t) = \frac{2tx(t)}{1 + x(t)}$$

$1 + x(t) \neq 0$
$x(t) = -1$

is given below.



Sketch the graph of the solutions to the initial value problems

(a) $x(0) = 1$

(b) $x(0) = -2$

(c) $x(0) = -0.5$

6. Match the direction field to the differential equations

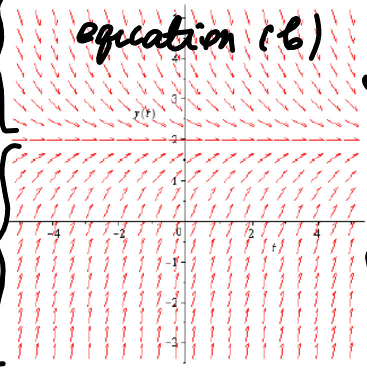
~~a) $y' = y - 2$~~ ~~b) $y' = 2 - y$~~ c) $y' = 2 + y$

d) $y' = -2 - y$ ~~e) $y' = (y-2)^2$~~ f) $y' = (y+2)^2$

$y'<0$

equation (b)

$y'>0$

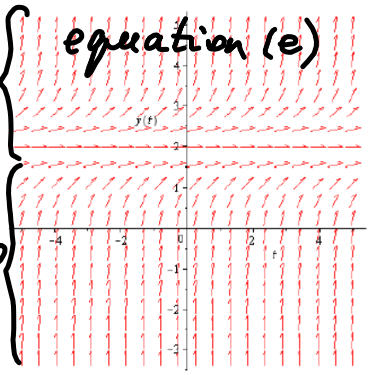


Direction field 1

$y'>0$

equation (e)

$y'>0$

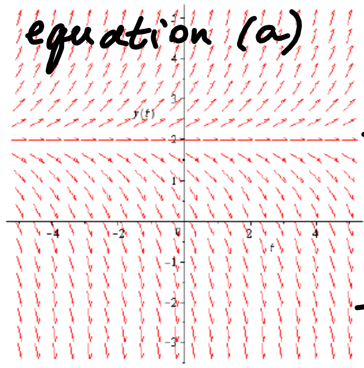


Direction field 2

equation (a)

$y'>0$

$y'<0$

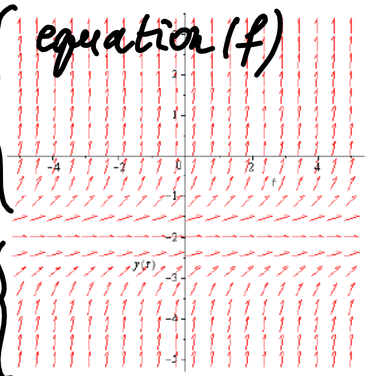


Direction field 3

$y'>0$

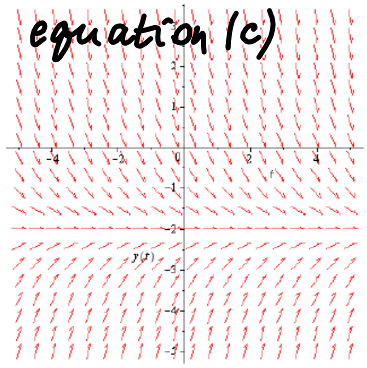
equation (f)

$y'>0$



Direction field 4

equation (c)

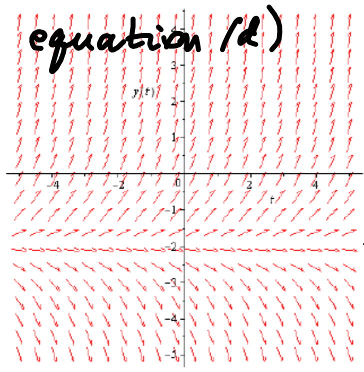


Direction field 5

equation (d)

$y'>0$

$y'<0$



Direction field 6

direction fields 1-3 slope = 0 when $y = 2$

4-6 slope = 0 when $y = -2$

7. Given the following differential equations, classify each as an ordinary differential equation, partial differential equation, give the order. If the equation is an ordinary differential equation, say whether the equation is linear or non linear.

(a) $\frac{dy}{dx} = 3y + x^2$.

ODE of order 1
linear

(b) $5\frac{d^4y}{dx^4} + y = x(x-1)$.

ODE of order 4
linear

(c) $\frac{\partial N}{\partial t} = \frac{\partial^2 N}{\partial r^2} + \frac{1}{r}\frac{\partial N}{\partial r} + kN$.

PDE of order 2

(d) $\frac{dy}{dx} = 3y + x^2$.

ODE of order 1, linear

(e) $\frac{dx}{dt} = x^2 - t.$ ODE of order 1
nonlinear

(f) $(1+y^2)y'' + ty' + y = e^t.$ ODE of order 2
nonlinear

(g) $\frac{dy}{dx} + xy^2 = 0.$ ODE of order 1
nonlinear

8. Verify that the functions $y_1(t) = t^{-2}$ and $y_2(t) = t^{-2}\ln(t)$ are solutions to the differential equation $t^2y'' + 5ty' + 4y = 0$.

Skip

9. (a) Show that $f(x) = (x^2 + Ax + B)e^{-x}$ is solution to

$$y'' + 2y' + y = 2e^{-x}$$

for all real numbers A and B.

$$f'(x) = (2x+A)e^{-x} - (x^2+Ax+B)e^{-x}$$

$$f''(x) = 2e^{-x} - (2x+A)e^{-x} - (2x+A)e^{-x} + (x^2+Ax+B)e^{-x}$$

Plug f, f', and f'' into the equation:

$$\underbrace{2e^{-x} - \cancel{2(2x+A)e^{-x}} + \cancel{(x^2+Ax+B)e^{-x}}}_{f''} + \underbrace{\cancel{2(2x+A)e^{-x}} - \cancel{2(x^2+Ax+B)e^{-x}}}_{2f'} + \underbrace{\cancel{(x^2+Ax+B)e^{-x}}}_{f}$$

$$= 2e^{-x}$$

(b) Find a solution that satisfies the initial condition $y(0) = 3$ and $y'(0) = 1$.

$y(x) = (x^2 + Ax + B)e^{-x}$

$y'(x) = (2x+A)e^{-x} - (x^2+Ax+B)e^{-x}$

$y(0) = 3$

$y'(0) = 1$

$y(0) = \boxed{B = 3}$

$y'(0) = A - B = 1$

$A = 1 + B$

$\boxed{A = 4}$

$$\boxed{y(x) = (x^2 + 4x + 3)e^{-x}}$$

10. Determine for which values of r the function t^r is a solution of the differential equation

$$t^2y'' - 4ty' + 4y = 0$$

$$y = t^r$$
$$y' = r t^{r-1}$$
$$y'' = r(r-1) t^{r-2}$$

Plug y, y', and y'' into the equation:

$$\underbrace{t^2 r(r-1) t^{r-2}}_{t^r} - \underbrace{4 t r t^{r-1}}_{t^r} + 4t^r = 0$$

$$r(r-1)\cancel{t^r} - 4r\cancel{t^r} + 4\cancel{t^r} = 0$$

$$r^2 - r - 4r + 4 = 0$$
$$r^2 - 5r + 4 = 0$$
$$(r-4)(r-1) = 0$$
$$\boxed{r_1 = 1 \mid r_2 = 4}$$

11. For which values of r is the function $(x-1)e^{-rx}$ solution to $y'' - 6y' + 9y = 0$.

$$y(x) = (x-1)e^{-rx}$$

$$y' = e^{-rx} - r(x-1)e^{-rx}$$

$$= (1 - rx + r)e^{-rx}$$

$$y'' = -re^{-rx} - r(1 - rx + r)e^{-rx}$$

$$= (r^2x - r^2 - 2r)e^{-rx}$$

$$(r^2x - r^2 - 2r)\cancel{e^{-rx}} - 6(1 - rx + r)\cancel{e^{-rx}} + 9(x-1)\cancel{e^{-rx}} = 0$$

$$\underline{r^2x} - r^2 - 2r - 6 + \underline{6rx} - 6r + \underline{9x} - 9 = 0$$

$$x\underbrace{(r^2 + 6r + 9)}_{(r+3)^2} \underbrace{- r^2 - 8r - 15}_{-(r+3)(r+5)} = 0$$

$$x(r+3)^2 - (r+3)(r+5) = 0$$

$$(r+3)\big(x(r+3) - (r+5)\big) = 0$$

$$r + 3 = 0 \quad \text{or} \quad \cancel{x(r+3) - (r+5)} = 0 \quad \text{NOT VALID}$$

should be a solution for all x

$$\boxed{r = -3}$$