1. Find the general solution of the given differential equation.

(a) $y' + 2ty = 2te^{-t^2}$.

LINEAR, $p(t)=2t$, $q(t)=2te^{-t^2}$

1) Integrating factor $\mu(t)$: $\frac{d\mu}{dt} - p(t)\mu = 0$

$$\frac{d\mu}{dt} - 2t\mu = 0$$

separate variables:

$$\int \frac{d\mu}{\mu} = \int 2t\,dt$$

$$\ln|\mu| = t^2$$

$$\mu(t) = e^{t^2}$$

2) $\frac{d}{dt}[\mu(t)y(t)] = \mu(t)q(t)$

$$\mu(t)y(t) = \int \mu(t)q(t)\,dt$$

$$e^{t^2}y(t) = \int e^{t^2}\,2te^{-t^2}\,dt$$

$$= \int 2t\,dt$$

$$e^{t^2}y(t) = t^2 + C$$

solve for $\boxed{y(t) = e^{-t^2}(t^2 + c)}$

(b) $y' = \dfrac{3x^2 - 1}{3 + 2y}$. separable

$$y' = \frac{dy}{dx} = \frac{3x^2-1}{3+2y}$$

$$\int (3+2y)\,dy = \int (3x^2-1)\,dx$$

$$\boxed{3y + y^2 = x^3 - x + C}$$ implicit solution

(c) $y' = 2x \sec y$. separable

$$\frac{dy}{dx} = 2x \sec y$$

$$\frac{dy}{\sec y} = 2x\,dx$$

$$\sec y = \frac{1}{\cos y}$$

$$\int \cos y \, dy = \int 2x\,dx$$

$$\boxed{\sin y = x^2 + C}$$ implicit solution

(d) $\underbrace{ty' + y = 3t\cos t}_{t}$, $t > 0$. linear

standart form:

$$y' + \frac{1}{t}y = 3\cos t$$

$$p(t) = \frac{1}{t},\ q(t) = 3\cos t$$

Integrating factor $\mu(t)$:

$$\frac{d\mu}{dt} - \frac{1}{t}\mu = 0$$

$$\frac{d\mu}{dt} = \frac{\mu}{t}$$

$$\int \frac{d\mu}{\mu} = \int \frac{dt}{t}$$

$$\ln|\mu| = \ln|t|$$

$$\boxed{\mu(t) = t}$$

$$\mu(t)y(t) = \int \mu(t)q(t)dt$$

$$t\,y(t) = \int 3t\cos t\,dt$$

integrate by parts

$$\int u\,dv = uv - \int v\,du$$

$u = t$ $\quad dv = \cos t\,dt$

$du = dt$ $\quad v = \sin t$

$$t\,y(t) = 3\left(t\sin t - \int \sin t\,dt\right)$$

$$t\,y(t) = 3t\sin t + 3\cos t + C$$

$$\boxed{y(t) = 3\sin t + \frac{3\cos t}{t} + \frac{C}{t}}$$

2. Find the solution to the initial value problem

(a) $$\frac{dy}{dx} = 4x^3y - y, \qquad y(1) = -3.$$

$\frac{dy}{dx} = y(4x^3-1)$ separable

$\int \frac{dy}{y} = \int (4x^3-1)\,dx$

$\ln|y| = x^4 - x + C$

$y(x) = e^{x^4-x+C} = e^{x^4-x} \cdot e^{C} = c_1$

$= C e^{x^4-x}$

Plug $y(x)$ into $y(1) = -3$

$-3 = y(1) = Ce^{1-1}$

$-3 = C$

$\boxed{y(x) = -3e^{x^4-x}}$

(b) $\dfrac{dy}{dx} + \dfrac{2y}{t} = \dfrac{\cos t}{t^2}, \quad y(1) = \dfrac{1}{2}, \qquad t > 0.$

$p(t) = \dfrac{2}{t}, \quad q(t) = \dfrac{\cos t}{t^2}$

Integrating factor $\mu(t)$:

$$\frac{d\mu}{dt} - \frac{2\mu}{t} = 0$$

$$\frac{d\mu}{dt} = \frac{2\mu}{t}$$

$$\int \frac{d\mu}{\mu} = \int \frac{2\,dt}{t}$$

$$\boxed{\mu(t) = t^2}$$

$$t^2 y(t) = \int \cancel{t^2} \frac{\cos t}{\cancel{t^2}}\, dt$$

$$t^2 y(t) = \sin t + C$$

$$y(t) = \frac{\sin t}{t^2} + \frac{C}{t^2}$$

$$1/2 = y(1) = \sin 1 + C$$

$$C = \frac{1}{2} - \sin 1$$

$$\boxed{y(t) = \frac{\sin t}{t^2} + \frac{1}{t^2}\left(\frac{1}{2} - \sin 1\right)}$$

(c) $2\sqrt{x}\dfrac{dy}{dx} = \cos^2 y, \qquad y(4) = \dfrac{\pi}{4}.$

$$\int \frac{dy}{\cos^2 y} = \int \frac{dx}{2\sqrt{x}}$$

$$\tan y = \frac{1}{2}\,\frac{x^{1/2}}{1/2} + C$$

$\tan y = x^{1/2} + C$ implicit solution.

Plug $x=4$ and $y=\frac{\pi}{4}$ into the equation:

$$\tan\frac{\pi}{4} = 4^{1/2} + C$$

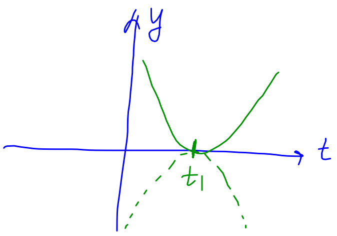
$$C = -1$$

$$\boxed{\tan y = x^{1/2} - 1}$$

3. Consider the initial value problem

$$y' + 2y = 5 - t, \qquad y(0) = y_0$$

Find the value y_0 for which the solution touches, but does not cross the t-axis.



$y(t)$ is a solution of the equation.
$y(t)$ touches the t-axis at t_1

$y(t_1) = 0$

$y'(t_1) = 0$ (tangent is horizontal, slope $= 0$)

$$\underbrace{y'(t_1)}_{0} + 2\underbrace{y(t_1)}_{0} = 5 - t_1$$

$5 - t_1 = 0$ OR $t_1 = 5$

The solution of $y' + 2y = 5 - t$, $y(5) = 0$

touches, but does not intersect t-axis.

Solve IVP:

$y' + 2y = 5 - t$

Integrating factor $\mu(t)$:

$$\frac{d\mu}{dt} - 2\mu = 0$$

$$\frac{d\mu}{\mu} = 2dt$$

$$\mu(t) = e^{2t}$$

$$e^{2t} y(t) = \int (5-t)e^{2t}\,dt \quad \text{by parts}$$

$u = 5 - t$, $dv = e^{2t}dt$

$du = -dt$, $v = \frac{1}{2}e^{2t}$

$$= (5-t)\frac{1}{2}e^{2t} - \int \frac{1}{2}e^{2t}(-dt)$$

$$e^{2t}y(t) = \left(\frac{5}{2} - \frac{t}{2}\right)e^{2t} + \frac{1}{4}e^{2t} + C$$

$$y(t) = \frac{11}{4} - \frac{t}{2} + Ce^{-2t}$$

$$0 = y(5) = \overbrace{\frac{11}{4} - \frac{5}{2}}^{1/4} + Ce^{-10}$$

$$C = -\frac{e^{10}}{4}$$

$$y(t) = \frac{11}{4} - \frac{t}{2} - \frac{e^{10}}{4}e^{-2t}$$

find the y-intercept:

$$y_0 = y(0) = \frac{11}{4} - \frac{e^{10}}{4}$$

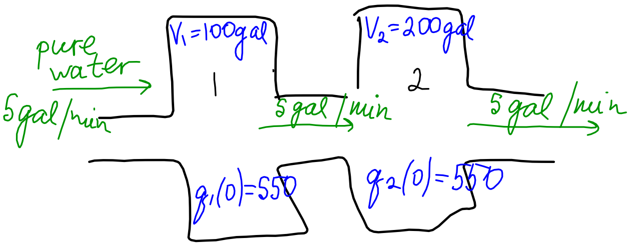
$$\boxed{y_0 = \frac{11 - e^{10}}{4}}$$

4. Consider a cascade of 2 tanks, with $V_1 = 100$ gal and $V_2 = 200$ gal the volume of brine of the 2 tanks. Each tank also initially contains 550lb of salt. Pure water flows into the first tank at a rate of 5 gal/min. The mixture flows from the first tank to the second one and flows out of the second tank at the same rate (5 gal/min).

Find the amount of salt in the 2 tanks at any time t.

$q_1(t)$ is the mass of salt in tank 1

$q_2(t)$ is the mass of salt in tank 2



Tank 1

$$\frac{dq_1}{dt} = \underbrace{\boxed{\text{rate in}}}_{=0} - \underbrace{\boxed{\text{rate out}}}_{=(\text{concentration})(\text{speed}) = \frac{q_1(t)}{100}\cdot 5}$$

$$\boxed{\frac{dq_1}{dt} = -\frac{q_1}{20}, \quad q_1(0) = 550} \qquad \boxed{q_1(t) = 550 e^{-\frac{t}{20}}}$$

Tank 2.

$$\frac{dq_2}{dt} = \boxed{\text{rate in}} - \boxed{\text{rate out}}$$

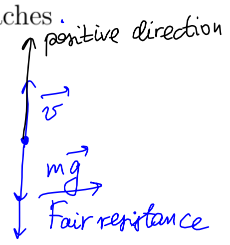
$$\frac{dq_2}{dt} = \frac{q_1(t)}{100}\cdot 5 - \frac{q_2(t)}{200}\cdot 5$$

$$\frac{dq_2}{dt} = \frac{q_1(t)}{20} - \frac{q_2(t)}{40}$$

$\mu(t) = e^{\frac{t}{40}}$ $$\boxed{\frac{dq_2}{dt} + \frac{q_2(t)}{40} = \frac{55}{2} e^{-\frac{t}{20}}, \quad q_2(0) = 550}$$

$$\boxed{q_2(t) = -110 e^{-\frac{t}{20}} + 660 e^{-\frac{t}{40}}}$$

5. A ball with mass 1kg is thrown upward with initial velocity 20 m/s from the roof of a building 50 m high. A force due to the resistance of the air of $v/10$, where the velocity is measured in m/s, acts on the ball. Find the maximum height above the ground that the ball reaches.



$$\text{net force} = m\vec{g} + \vec{F}_{air\ resistance}$$

$$m\vec{a} = m\vec{g} + \vec{F}_{air\ resistance}$$

$$m\frac{dv}{dt} = -mg - \frac{v}{10}, \quad g = 9.81\ m/sec^2$$

$$\boxed{\frac{dv}{dt} = -g - \frac{v}{10}, \quad v(0) = 20}$$

$$\frac{dv}{g + \frac{v}{10}} = -dt$$

$$10 \ln\left|g + \frac{v}{10}\right| = -t + C$$

$$\ln\left|g + \frac{v}{10}\right| = -\frac{t + C}{10}$$

$$g + \frac{v}{10} = e^{-\frac{t}{10}} \cdot \underbrace{e^{-\frac{C}{10}}}_{= C_1}$$

$$v = -10g + 10C_1 e^{-\frac{t}{10}}$$

$$20 = v(0) = -10g + 10C_1$$

$$C_1 = 2 + g$$

$$\boxed{v(t) = -10g + 10(2+g)e^{-\frac{t}{10}}}$$

Find t such that $v(t) = 0$

$$-\cancel{10}g + \cancel{10}(2+g)e^{-\frac{t}{10}} = 0$$

$$(2+g)e^{-\frac{t}{10}} = g$$

$$\boxed{t_{max} = -10 \ln \frac{g}{2+g}}$$

$x(t)$ - height of the object at time t.

$$x(t) = \int v(t)\,dt$$

$$= -10gt - 100(2+g)e^{-\frac{t}{10}} + C$$

$$x(0) = 50$$

$$C = 50 + 100(2+g)$$

$$= 250 + 100g$$

$$\boxed{x(t) = 250 + 100g - 10gt - 100(2+g)e^{-\frac{t}{10}}}$$

$$\boxed{x_{max} = x(t_{max})}$$

6. College graduate borrows \$10,000 to buy a car. The lender charges interest at an annual rate of 10%. Assuming that the interest is compounded continuously and that the borrows makes payment continuously at a constant annual rate k, determine the payment rate k that is required to pay off the loan in 5 years. Also determine how much interest is paid during the 5-year period.

$S(t)$ is the balance of the loan at time t.

$$\frac{dS}{dt} = rS - k$$

r is the annual rate

$$r = 10\% = 0.1$$

k is the monthly payment rate. $\boxed{k - ?}$

$$S(0) = 10000, \quad S(5) = 0$$

$$\frac{dS}{dt} = 0.1S - k, \quad S(0) = 10000, \; S(5) = 0$$

$$\frac{dS}{0.1S - k} = dt$$

$$10 \ln|0.1S - k| = t + C$$

$$\ln|0.1S - k| = \frac{t}{10} + \frac{C}{10}$$

$$0.1S - k = Ce^{\frac{t}{10}}$$

$$S = 10\left(k + Ce^{\frac{t}{10}}\right)$$

$$10000 = S(0) = 10k + 10C$$

$$C = 1000 - k$$

$$S(t) = 10k + (1000 - k)e^{\frac{t}{10}}$$

$$0 = S(5) = 10k + (1000 - k)e^{1/2}$$

$$10k + (1000 - k)1.65 = 0$$

$$k = -\frac{1000}{8.35}$$

postponed to WIR#3

7. Food, initially at a temperature of 40°F, was placed in an oven preheated to 350°F. After 10 minutes in the oven, the food had warmed to 120°F. After 20 minutes, the food was removed from the oven and allowed to cool at room temperature (72°F). If the ideal serving temperature is 110°F, when should the food be served?

$T(t)$ is the temperature of food at time t.

$$T(0) = 40.$$

outside temperature $M(t) = 350$

$$\frac{dT}{dt} = k(350 - T(t)), \quad T(10) = 120$$

$$\frac{dT}{350 - T(t)} = k\,dt$$

$$-\ln|350 - T(t)| = kt + C$$

$$350 - T(t) = Ce^{-kt}$$

$$T(t) = 350 - Ce^{-kt}$$

$$T(0) = 350 - C = 40$$

$$C = 310$$

$$T(t) = 350 - 310e^{-kt}$$

$$120 = T(10) = 350 - 310e^{-10k}$$

$$e^{-10k} = \frac{230}{310}$$

$$k = -\frac{1}{10}\ln\frac{230}{310} \approx 0.03$$

$$T(t) = 350 - 310e^{-0.03t}$$

Find t such that $T(t) = 110$

$$110 = 350 - 310e^{-0.03t}$$

$$t = -\frac{1}{0.03}\ln\frac{240}{310} \approx \boxed{8.5 \text{ (min)}}$$