1. College graduate borrows \$10,000 to buy a car. The lender charges interest at an annual rate of 10%. Assuming that the interest is compounded continuously and that the borrows makes payment continuously at a constant annual rate k, determine the payment rate k that is required to pay off the loan in 5 years. Also determine how much interest is paid during the 5-year period.

S(t) is the balance of the loan at time t

k - is the constant payment annual rate

$S(0) = 10{,}000$

$r = 0.1 = 10\%$

Equation for S(t):

$$\frac{dS}{dt} = rS - 5k, \quad S(0) = 10{,}000$$

$$\boxed{\frac{dS}{dt} = 0.1S - 5k, \quad S(0) = 10{,}000}$$ IVP for S(t).

$$\frac{dS}{dt} = 0.1(S - 50k)$$

$$\int \frac{dS}{S - 50k} = \int 0.1\,dt$$

$$\ln|S - 50k| = 0.1t + C$$

$$S - 50k = Ce^{0.1t}$$

$$S(t) = 50k + Ce^{0.1t}$$

$$S(0) = 50k + C = 10000$$

$$C = 10000 - 50k$$

$$S(t) = 50k + (10000 - 50k)e^{0.1t} \qquad S(5) = 0$$

$$S(5) = 50k + (10000 - 50k)e^{0.5} = 0$$

solve for k:

$$50k + 10000e^{0.5} - 50ke^{0.5} = 0$$

$$50k(1 - e^{0.5}) = -10000e^{0.5}$$

$$k = -\frac{10000e^{0.5}}{50(1 - e^{0.5})} \approx -\frac{16487.21}{-32.44} \approx 508.30$$

2. Food, initially at a temperature of 40°F, was placed in an oven preheated to 350°F. After 10 minutes in the oven, the food had warmed to 120°F. After 20 minutes, the food was removed from the oven and allowed to cool at room temperature (72°F). If the ideal serving temperature is 110°F, when should the food be served?

$T(t)$ is the temperature of food at time t.

$T(0)=40$

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in the oven

outside temperature $= 350$

$T(10)=120$

$$\frac{dT}{dt}=k\,(350-T(t))$$

$$T(t)=350-310\,e^{-0.03t} \quad \text{(in the oven)}$$

$$T(20)=350-310\,e^{-0.6}\approx 180^\circ$$

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in the room.

$T(0)=180$

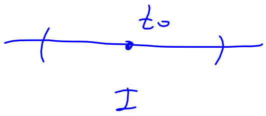
outside temperature $= 72$

$$\frac{dT}{dt}=k_1\,(72-T(t))$$

Find t_1 such $T(t_1)=110$.

. . . .

Initial value Problem.



(*) $$\begin{aligned} &y'+p(t)y=g(t)\\ &y(t_0)=y_0 \end{aligned}$$ – linear

If both $p(t)$ and $g(t)$ are continuous on some interval I containing t_0, then the initial value problem (*) has a unique solution.

(**) $$\begin{cases} y'=f(t,y)\\ y(t_0)=y_0 \end{cases}$$ – nonlinear

If both $f(t,y)$ and $\frac{\partial f}{\partial y}$ are continuous in some rectangle $\alpha \le t \le \beta, \gamma \le y \le \delta$ containing (t_0, y_0), then the initial value problem (**) has a unique solution in the interval $t_0-h < t < t_0+h$

3. Determine an interval in which the solutions of the following initial value problems are certain to exist.

(a) $y' + \frac{\sin t}{t^2-1} y = \frac{\cot t}{t^2-4t+3}, \qquad y(2) = -1.$

linear

$$p(t) = \frac{\sin t}{t^2-1}$$

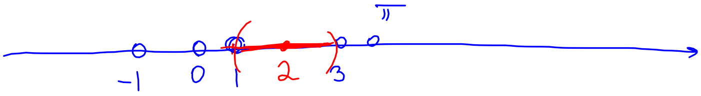
$$g(t) = \frac{\cot t}{t^2-4t+3}$$

$$= \frac{\cot t}{(t-3)(t-1)}$$

$p(t)$ is continuous on $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$

$g(t)$ is discontinuous at:

$t=3$, $t=1$, $t=\pi n$, $n=0,\pm1,\pm2,..$



$$\boxed{(1,3)}$$

(b) $t(t-4)y' + t^2 \ln(t+5)y = 0, \qquad y(-3) = 7.$

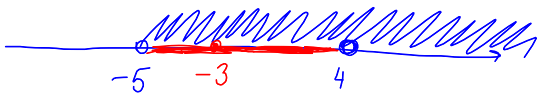
linear

Standart form: $y' + \frac{t^{\cancel{2}}\ln(t+5)}{\cancel{t}(t-4)}y = 0$

$$y' + \frac{t\ln(t+5)}{t-4}y = 0$$

$$p(t) = \frac{t\ln(t+5)}{t-4}$$

p(t) is defined when $t+5>0$
$t \neq 4$



p(t) is continuous on $(-5,4)\cup(4,\infty)$

$$\boxed{(-5,4)}$$

4. State where in the ty-plane the hypothesis of theorem 2.4.2 are satisfied.

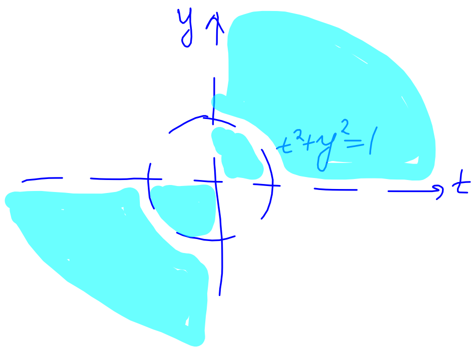
(a) $y' = \frac{\ln(ty)}{1-(t^2+y^2)}$.   nonlinear

$f(t,y) = \frac{\ln(ty)}{1-(t^2+y^2)}$ continuous if $\begin{pmatrix} ty>0 \\ t^2+y^2 \neq 1 \end{pmatrix}$

$\frac{\partial f}{\partial y} = \frac{\frac{1}{y}(1-t^2-y^2)-(-2y)\ln(ty)}{[1-(t^2+y^2)]^2}$ continuous if

$ty>0$

$t^2+y^2 \neq 1$



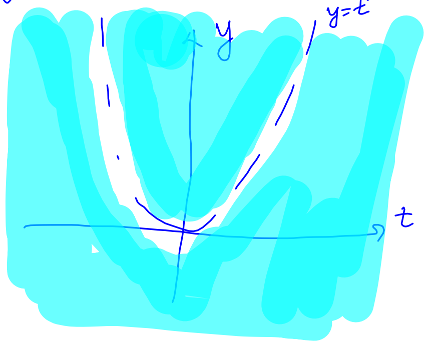
Points in the 1st and the 3rd quadrants, outside the circle $t^2+y^2=1$ and the coordinate axes.

(b) $y' = (t^2 - y)^{1/3}$.

$f(t,y) = (t^2-y)^{1/3}$ continuous for all t and y

$\frac{\partial f}{\partial y} = \frac{1}{3}(t^2-y)^{-2/3}(-1)$ continuous if $t^2-y \neq 0$

or $y \neq t^2$



$\boxed{\text{all } t \text{ and } y \text{ outside the parabola } y = t^2.}$

5. Solve the following initial value problems and determine how the interval in which the solution exists depends on the initial value t_0.

(a) $y' = \frac{-4}{t}y, \qquad y(t_0) = y_0$

linear

separable

$p(t) = -\frac{4}{t}$ should be continuous.

$t \neq 0$

$$\frac{dy}{dt} = -\frac{4y}{t}$$

$$\int \frac{dy}{y} = -\int \frac{4\,dt}{t}$$

$$\ln|y| = -4\ln|t| + C$$

$$\ln|y| = \underbrace{\ln|t^{-4}| + \ln C_1}$$

$$\ln|y| = \ln|C_1 t^{-4}|$$

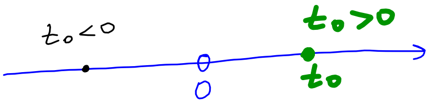
$$y(t) = C_1 t^{-4}$$

$$y(t_0) = y_0$$

$$y(t_0) = C_1 t_0^{-4} = y_0$$

$$C_1 = \frac{y_0}{t_0^{-4}} = y_0 t_0^4$$

$$\boxed{y(t) = \frac{y_0 t_0^4}{t^4}}$$ solution to the initial value problem.



if $t_0>0$, then the solution exists on $(0,\infty)$

if $t_0<0$, then the solution exists on $(-\infty,0)$

(b) $y' + y^3 = 0 \qquad y(t_0) = y_0.$

nonlinear

$f(t,y) = -y^3$ continuous for all y

$\frac{\partial f}{\partial y} = -3y^2$ continuous for all y.

$$\frac{dy}{dt} = -y^3$$

$$\int \frac{dy}{y^3} = -\int dt$$

$$\frac{y^{-2}}{-2} = -t + C$$

solve for y.

$$y^{-2} = 2(t-C)$$

$$y^2 = \frac{1}{2(t-C)}$$

$$y = \sqrt{\frac{1}{2t-2C}}$$

$$y(t_0) = \underbrace{y_0 = \sqrt{\frac{1}{2t_0-2C}}}$$ solve for C.

$$y_0^2 = \frac{1}{2t_0-2C}$$

$$C = t_0 - \frac{1}{2y_0^2}$$

Solution of the initial value problem:

$$\boxed{y(t) = \sqrt{\frac{1}{2t-2\left(t_0-\frac{1}{2y_0^2}\right)}}}$$

$y_0 \neq 0$

$$2t - 2\left(t_0 - \frac{1}{2y_0^2}\right) > 0$$

$$t > t_0 - \frac{1}{2y_0^2}$$

$y_0 = 0$

$y' = -y^3,\ y(t_0) = 0$

solution $y(t) = 0$

6. Verify that both $y_1 = 1 - t$ and $y_2 = \frac{-t^2}{4}$ are solutions to the same initial value problem

$$y'(t) = \frac{-t + (t^2 + 4y)^{(1/2)}}{2}, \qquad y(\overset{t}{2}) = \overset{y}{-1}.$$

Does it contradict the existence and uniqueness theorem?

$y_1 = 1 - t$ $\qquad$ $y_1(2) = 1 - 2 = -1$ (satisfies the initial condition)

$y_1' = -1$

Plug y_1 and y_1' into the equation:

$$-1 = \frac{-t + (t^2 + 4(1-t))^{1/2}}{2}$$

$$-1 = \frac{-t + (t^2 + 4 - 4t)^{1/2}}{2} \qquad \boxed{t^2 - 4t + 4 = (t-2)^2}$$

$$-1 = \frac{-t + [(t-2)^2]^{1/2}}{2}$$

$$-1 = \frac{-t + t - 2}{2}$$

$$\boxed{-1 = -1}$$

$y_2 = -\frac{t^2}{4}$; $\quad y_2(2) = \frac{-4}{4} = -1$ satisfies the initial condition

$y_2' = -\frac{2t}{4} = -\frac{t}{2}$

Plug y_2 and y_2' into the equation:

$$-\frac{t}{2} = \frac{-t + (t^2 + \overbrace{4(-\frac{t^2}{4})}^{0})^{1/2}}{2}$$

$$-\frac{t}{2} = -\frac{t}{2}$$

$$f(t,y) = \frac{-t + (t^2+4y)^{1/2}}{2}$$

$t^2 + 4y \geq 0$

$\underbrace{2}_{t_0}{}^2 + 4\underbrace{(-1)}_{y_0} = 0.$

$$\frac{\partial f}{\partial y} = \frac{1}{2}\frac{1}{2}(t^2 + 4y)^{-1/2}(4)$$

$= (t^2+4y)^{-1/2}$ $\qquad$ continuous if $t^2 + 4y > 0$

$\frac{\partial f}{\partial y}$ is discontinuous at $(2,-1)$.

7. Given the differential equation

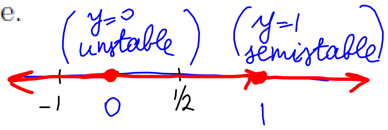
$$y'(t) = y^3 - 2y^2 + y$$

(a) Find the equilibrium solutions

$$y^3 - 2y^2 + y = 0$$
$$y(y^2 - 2y + 1) = 0$$
$$y(y-1)^2 = 0$$

Equilibrium solutions: $y_0 = 0$ and $y_1 = 1$

(b) Graph the phase line. Classify each equilibrium solution as either stable, unstable, semistable.



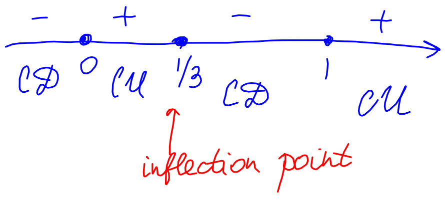
$f(-1) = (-1)^3 - 2(-1)^2 - 1 < 0$

$f(1/2) = \frac{1}{8} - 2\frac{1}{4} + \frac{1}{2} = \frac{1}{8} > 0$

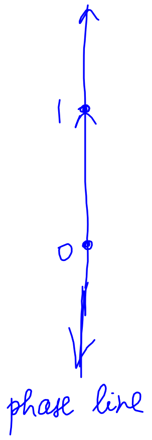
$f(2) = 8 - 8 + 2 = 2 > 0$

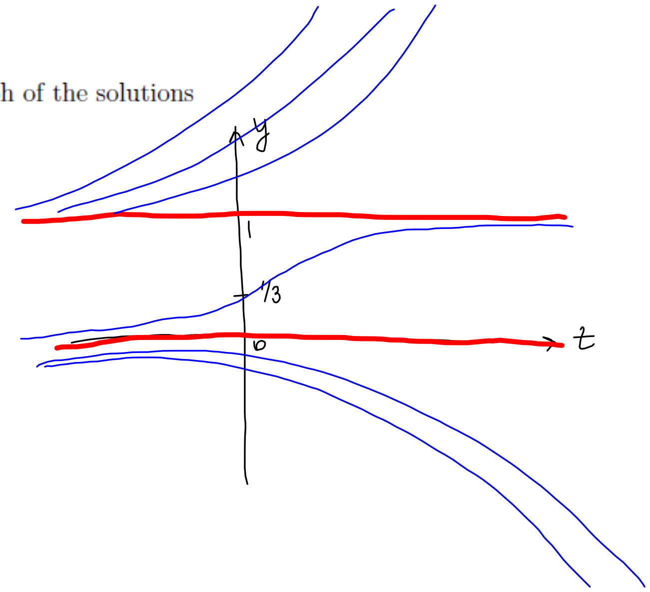
(c) Determine where the solutions are concave up? concave down?

$$(y')' = (y^3 - 2y^2 + y)'$$
$$y'' = 3y^2y' - 4yy' + y'$$
$$= (3y^2 - 4y + 1)y'$$
$$= \underbrace{(3y^2 - 4y + 1)}_{(3y-1)(y-1)}\underbrace{(y^3 - 2y^2 + y)}_{y(y-1)^2}$$
$$= y(3y-1)(y-1)^3$$



(d) Sketch the graph of the solutions





8. Given the differential equation $y' = y(y-2)$

(a) Sketch the graph of f versus y.

(b) Find the equilibrium solutions

$y=0$ stable

$y=2$ unstable

(c) Graph the phase line. Classify each equilibrium solution as either stable, unstable, semistable.

(d) Determine where the solutions are concave up? concave down?

(e) Sketch the graph of the solutions