Math 308 WEEK in REVIEW 5 Spring 2014

1. Find the solution to the given initial value problem.

(a) $y'' + 10y' + 25y = 0, \quad y(0) = 2, \quad y'(0) = -1.$

auxiliary eqn: $(r+5)^2 = 0$

$r^2 + 10r + 25 = 0$

$r_1 = \frac{-10 + \sqrt{100 - 4(25)}}{2} = -5$ - repeated root

General solution:

$y(t) = (c_1 + c_2 t)e^{-5t}$

plug $y(t)$ into initial conditions:

$2 = y(0) = c_1 \qquad \boxed{c_1 = 2}$

$y'(t) = c_2 e^{-5t} - 5(c_1 + c_2 t)e^{-5t}$

$-1 = y'(0) = c_2 - 5c_1$

$c_2 = -1 + 5c_1$

$= \boxed{9 = c_2}$

Solution of IVP: $\boxed{y(t) = (2 + 9t)e^{-5t}}$

(b) $y'' + 9y = 0, \qquad y(0) = -2, \quad y'(0) = 3.$

auxiliary eqn: $r^2 + 9 = 0$

$r^2 = -9$

$r = \pm\sqrt{-9}$

$= \pm 3i, \quad r_1 = 3i, \quad r_2 = -3i = \overline{r_1}$

$Re(r_1) = 0, \quad Im(r_1) = 3$

General solution

$$y(t) = \left[C_1 \cos(3t) + C_2 \sin(3t)\right] e^{0 \cdot t}$$

$-2 = y(0) = C_1 \qquad \boxed{C_1 = -2}$

$y'(t) = -3C_1 \sin(3t) + 3C_2 \cos(3t)$

$3 = y'(0) = 3C_2$

$\boxed{C_2 = 1}$

Solution of IVP

$$\boxed{y(t) = -2\cos(3t) + \sin(3t)}$$

(c) $y'' - 2y' + 5y = 0 \qquad y(\pi/2) = 0, \quad y'(\pi/2) = 2.$

auxiliary eqn. $\quad r^2 - 2r + 5 = 0$

$$r_1 = \frac{2 + \sqrt{4 - 5(4)}}{2}$$

$$= \frac{2 + \sqrt{-16}}{2} \qquad \sqrt{-16} = 4i$$

$$= 1 + 2i$$

$Re(r_1) = 1$

$Im(r_1) = 2$

General solution: $y(t) = e^{1 \cdot t}\left(c_1 \cos(2t) + c_2 \sin(2t)\right)$

2. Use the method of reduction of order to find a fundamental set of solutions.

(a) $t^2y'' + 2ty' - 2y = 0, \quad t > 0, \quad y_1(t) = t.$

$y'' + p(t)y' + q(t)y = 0$

Given a solution $y_1(t)$ of the equation. Find a second solution.

We look for the general solution of the eqn in the form

$y(t) = v(t)y_1(t)$, where $v(t)$ is an unknown function.

Plug y, y', y'' into the equation, and then we'll get an equation for v.

$t^2y'' + 2ty' - 2y = 0, \quad t > 0$

$y_1(t) = t$

$y(t) = t\,v(t)$

$v(t)$ is an unknown function

$y'(t) = v(t) + t\,v'(t)$

$y''(t) = v'(t) + v'(t) + t\,v''(t)$

$= 2v'(t) + t\,v''(t)$

Plug $y(t), y'(t), y''(t)$ into the eqn.

$$t^2\underbrace{\left(2v'(t) + t\,v''(t)\right)}_{y''(t)} + 2t\underbrace{\left(\cancel{v(t)} + t\,v'(t)\right)}_{y'(t)} - 2\underbrace{t\,\cancel{v(t)}}_{y(t)} = 0$$

$$\frac{t^3v''(t) + 4t^2v'(t) = 0}{t^3}$$

$$v'' + \frac{4}{t}v' = 0$$

substitution $\quad w = v'$

$w' = v''$

$w' + \frac{4}{t}w = 0$ first-order eqn

$$\frac{dw}{dt} = -\frac{4w}{t}$$

$$\int \frac{dw}{w} = -\int \frac{4\,dt}{t}$$

$\ln|w| = -4\ln|t| + \ln C_1$

$v'(t) = w(t) = C_1 t^{-4}$

$v(t) = C_1\int t^{-4}\,dt$

$= C_1\frac{t^{-3}}{-3} + C_2$

$= C_3 t^{-3} + C_2, \quad C_3 = \frac{C_1}{-3}$ new constant

$\boxed{v(t) = C_3 t^{-3} + C_2}$

Plug $v(t)$ into $y(t)$:

$y(t) = t\,v(t)$

$= t(C_3t^{-3} + C_2)$

$$= C_3\underbrace{t^{-2}}_{y_2(t)} + C_2\underbrace{t}_{y_1(t)}$$

$\boxed{y_2(t) = t^{-2}}$

(b) $(t-1)y'' - ty' + y = 0, \qquad t > 0, \quad y_1(t) = e^t.$

$$y(t) = e^t v(t)$$
$$y'(t) = e^t v(t) + e^t v'(t)$$
$$y''(t) = e^t v(t) + 2e^t v'(t) + e^t v''(t)$$

Plug y, y', y'' into the eqn:

$$(t-1)\cancel{e^t}\left(v(t)+2v'(t)+v''(t)\right) - t\cancel{e^t}\left(v(t)+v'(t)\right)+\cancel{e^t}v(t)=0$$

$$\cancel{(t-1)v(t)}+2(t-1)v'(t)+(t-1)v''(t)-\cancel{tv(t)}-tv'(t)+\cancel{v(t)}=0$$

$$(t-1)v''(t)+(t-2)v'(t)=0$$

Substitution $v'=w$, then $v''=w'$

$$w' = -\frac{t-2}{t-1}w$$

$$\frac{dw}{dt} = -\frac{t-2}{t-1}w$$

$$\int\frac{dw}{w} = -\int\frac{t-2}{t-1}dt \qquad \frac{t-2}{t-1} = \frac{t-1}{t-1} - \frac{1}{t-1} = 1 - \frac{1}{t-1}$$

$$e^{\ln|w|} = e^{-t+\ln|t-1|+\ln C_1}$$

$$v'(t) = w(t) = (t-1)e^{-t}C_1$$

$$v(t) = C_1\int(t-1)e^{-t}dt \qquad \text{by parts } \boxed{\int u\,dv = uv - \int v\,du}$$

$$u = t-1 \qquad dv = e^{-t}$$
$$du = dt \qquad v = -e^{-t}$$

$$= C_1\left(-e^{-t}(t-1) - \int(-e^{-t})\,dt\right)$$
$$= C_1\left(-e^{-t}(t-1) - e^{-t} + C_2\right)$$
$$= C_1\left(-te^{-t} + C_2\right)$$
$$= C_3te^{-t} + C_4 \qquad (C_3 = -C_1,\ C_4 = C_1C_2)$$

Plug $v(t)$ into $y(t)$:

$$y(t) = e^t v(t)$$
$$= e^t\left(C_3te^{-t} + C_4\right)$$
$$= C_3\underbrace{t}_{y_2(t)} + C_4\underbrace{e^t}_{y_1(t)}$$

$$\boxed{y_2(t) = t}$$

3. Verify that the functions y_1 and y_2 are solutions of the given differential equation. Do they constitute a fundamental set of solution

(a) $x^2y'' - x(x+2)y' + (x+2)y = 0,\quad x > 0 \quad y_1(x) = x,\quad y_2(x) = xe^x.$

$\{y_1(t), y_2(t)\}$ form the fundamental set of solutions if they are linearly independent

$W[y_1, y_2](t) \neq 0$ at least at one point.

$$W[y_1, y_2] = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

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if $y_1(t)$ and $y_2(t)$ are solutions of

$y'' + p(t)y' + q(t)y = 0,$

then $\boxed{W[y_1, y_2] = Ce^{-\int p(t)dt}}$ Abel's formula

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$y_1(x) = x, \quad y_2(x) = xe^x$

$y_1'(x) = 1, \quad y_2'(x) = e^x + xe^x$

$$W[x, xe^x] = \begin{vmatrix} x & xe^x \\ 1 & e^x + xe^x \end{vmatrix}$$

$$= xe^x + x^2e^x - xe^x$$

$$= x^2e^x \neq 0 \quad (\text{since } x > 0)$$

$\{x, xe^x\}$ constitute the fundamental set of solutions.

$\boxed{\text{YES}}$

(b) $y'' + 4y = 0, \qquad y_1 = 2\sin^{2} x^{\cancel{4}} - 1, \quad y_2 = 3\sin^2 x - \cos^2 x - 1.$

$$y_1 = 2\sin^2 x - 1 \qquad\qquad y_2 = 3\sin^2 x - \cos^2 x - 1$$

$$y_1' = 4\sin x\cos x \qquad\qquad y_2' = 6\sin x\cos x - 2\cos x(-\sin x)$$

$$= 8\sin x\cos x$$

$$W[y_1, y_2](x) = \begin{vmatrix} 2\sin^2 x - 1 & 3\sin^2 x - \cos^2 x - 1 \\ 4\sin x\cos x & 8\sin x\cos x \end{vmatrix}$$

$$= 16\sin^3 x\cos x - 8\sin x\cos x - 12\sin^3\cos x + 4\sin x\cos^3 x + 4\sin x\cos x$$

$$= 4\sin^3 x\cos x + 4\sin x\cos^3 x - 4\sin x\cos x$$

$$= 4\sin x\cos x\underbrace{(\sin^2 x + \cos^2 x)}_{1} - 4\sin x\cos x$$

$$= 0$$

$\boxed{\text{NO}}$

4. If the Wronskian of f and g is $3e^{4t}$ and $f(t) = e^{2t}$, find $g(t)$.

$$W[f,g] = 3e^{4t}$$

$$f(t) = e^{2t}$$

Find $g(t)$.

$$W[f,g] = \begin{vmatrix} e^{2t} & g(t) \\ 2e^{2t} & g'(t) \end{vmatrix}$$

$$= \frac{\left[g'(t)e^{2t} - 2e^{2t}g(t) = 3e^{4t} \right]}{e^{2t}}$$

$$g'(t) - 2g(t) = 3e^{2t}$$ linear, 1st order

integrating factor:

$$\frac{d\mu}{dt} = -2\mu$$

$$\int \frac{d\mu}{\mu} = -\int 2\,dt$$

$$\mu(t) = e^{-2t}$$

$$e^{-2t}g(t) = \int 3e^{2t}e^{-2t}\,dt$$

$$e^{-2t}g(t) = 3\int dt$$

$$e^{-2t}g(t) = 3t + C$$

$$\boxed{g(t) = 3te^{2t} + Ce^{2t}}$$

5. If the Wronskian of f and g is $t\cos t - \sin t$, and if $u = 2f - 3g$, and $v = f + g$, Find the Wronskian of u and v.

$$W[f,g] = t\cos t - \sin t$$

$$W[f,g] = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g$$

$$fg' - f'g = t\cos t - \sin t$$

$$u = 2f - 3g$$
$$v = f + g$$

$W[u,v] - ?$

$$u' = 2f' - 3g'$$
$$v' = f' + g'$$

$$W[u,v] = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$$

$$= \begin{vmatrix} 2f - 3g & f + g \\ 2f' - 3g' & f' + g' \end{vmatrix}$$

$$= (2f - 3g)(f' + g') - (f + g)(2f' - 3g')$$

$$= \cancel{2ff'} + 2fg' - 3gf' - \cancel{3gg'} - \cancel{2f'f} + 3g'f - 2f'g + \cancel{3gg'}$$

$$= 5fg' - 5gf'$$

$$= 5(fg' - gf')$$

$$= 5\,W[f,g]$$

$$= \boxed{5(t\cos t - \sin t)}$$

$$ay'' + by' + cy = g(t)$$

auxiliary eqn: $ar^2 + br + c = 0.$

Case 1. $g(t) = p_0 t^m + p_1 t^{m-1} + \dots + p_{m-1} t + p_m$

Then

- $y_p(t) = At^m + Bt^{m-1} + \dots + Ct + D$

  not a root of the corresponding aux. eqn

- $y_p(t) = t(At^m + Bt^{m-1} + \dots + Ct + D)$

  if $r=0$ is one of two roots of the auxiliary eqn.

Case 2. $g(t) = e^{at}(p_0 t^m + p_1 t^{m-1} + \dots + p_{m-1} t + p_m)$

Then

- $y_p(t) = e^{at}(At^m + Bt^{m-1} + \dots + Ct + D)$

  if $r=a$ is not a root of the corresponding aux. eqn

- $y_p(t) = te^{at}(At^m + Bt^{m-1} + \dots + Ct + D)$

  if $r=a$ is one of two roots of the auxiliary eqn

- $y_p(t) = t^2 e^{at}(At^m + Bt^{m-1} + \dots + Ct + D)$

  if $r=a$ is a repeated root of the aux. eqn.

6. Find the general solution to the following equations

(a) $y'' - y' = t$

$$y'' + p(t)y' + q(t)y = g(t)$$

General solution of the nonhomogeneous eqn

$$y(t) = y_h(t) + y_p(t),$$

where $y_h(t)$ is the general solution of the corresponding homogeneous eqn

$y_p(t)$ is a particular solution of the nonhomogeneous eqn.

$$y'' - y' = t$$

Corresponding homogeneous eqn: $y'' - y' = 0$

auxiliary eqn: $r^2 - r = 0$

$$r(r-1) = 0$$

$$r_1 = 0, \quad r_2 = 1$$

General solution of the homogeneous eqn is

$$y_h(t) = C_1 + C_2 e^t$$

Particular solution of the nonhomogeneous eqn.

$g(t) = t$

polynomial of degree 1

$$y_p(t) = t(At + B)$$

polynomial of degree 1 with unknown coefficients

$r = 0$ is a root of the auxiliary eqn

$$y_p(t) = At^2 + Bt$$

$$y_p' = 2At + B$$

$$y_p'' = 2A$$

Plug y_p, y_p', y_p'' into the nonhomogeneous eqn.

$$\underbrace{2A}_{y_p''} \underbrace{- 2At - B}_{-y_p'} = t$$

$$\left.\begin{array}{l} t: \ -2A = 1 \\ 1: \ 2A - B = 0 \end{array}\right\} \quad \boxed{A = -1/2} \quad \boxed{B = -1}$$

$$y_p(t) = t\left(-\frac{1}{2}t - 1\right)$$

General solution of the nonhomogeneous eqn:

$$\boxed{y(t) = C_1 + C_2 e^t + t\left(-\frac{1}{2}t - 1\right)}$$

(c) $y'' - 2y' - 3y = -3e^{-t}$.

$y'' - 2y' - 3y = 0$ corresponding homogeneous eqn.

$r^2 - 2r - 3 = 0$ - auxiliary eqn.

$(r-3)(r+1) = 0$

$r_1 = -1, \ r_2 = 3$

$y_h(t) = C_1 e^{-t} + C_2 e^{3t}$

$g(t) = -3e^{-t} \rightarrow a = -1$

constant = polynomial of degree 0.

$y_p(t) = tAe^{-t}$

$y_p' = Ae^{-t} - Ate^{-t}$

$y_p'' = -Ae^{-t} - Ae^{-t} + Ate^{-t}$

$= -2Ae^{-t} + Ate^{-t}$

Plug y_p, y_p', y_p'' into the eqn:

$$\underbrace{-2Ae^{-t} + Ate^{-t}}_{y_p''} - 2\underbrace{(Ae^{-t} - Ate^{-t})}_{y_p'} - 3\underbrace{\cancel{A}te^{-t}}_{y_p} = -3e^{-t}$$

$-4Ae^{-t} = -3e^{-t}$

$A = \frac{3}{4}$

$y_p(t) = \frac{3}{4}te^{-t}$

General solution: $\boxed{y(t) = C_1 e^{-t} + C_2 e^{3t} + \frac{3}{4}te^{-t}}$

(b) $y'' - 2y' - 3y = 3te^{2t}$.

$y'' - 2y' - 3y = 0$

$r^2 - 2r - 3 = 0$

$r_1 = -1, \ r_2 = 3$

$g(t) = 3t e^{2t}$ → $a = 2$ (not a root of the aux. eqn)

polynomial of degree 1

$y_p(t) = (At + B)e^{2t}$