$ay''+by'+cy=f(t)$

a,b,c are constants

$y(t)=\underbrace{y_h(t)}_{\text{general sol. of the corresponding homogeneous eqn.}}+\underbrace{y_p(t)}_{\text{a particular solution of the nonhomogeneous eqn}}$

Undetermined coefficients.

1) $f(t)=p_0t^m+p_1t^{m-1}+\dots+p_m$ (polynomial of degree m)

Then

- $y_p(t)=At^m+Bt^{m-1}+\dots+C$ (polynomial of degree m with unknow coefficients) if $r=0$ is not a root of the corresponding auxiliary eqn.
- $y_p(t)=t(At^m+Bt^{m-1}+\dots+C)$ if $r=0$ is one of two roots of the auxiliary eqn.

2) $f(t)=(p_0t^m+p_1t^{m-1}+\dots+p_m)e^{at}$

Then

- $y_p(t)=(At^m+Bt^{m-1}+\dots+C)e^{at}$ if $r=a$ is not a root of the auxiliary eqn
- $y_p(t)=t(At^m+Bt^{m-1}+\dots+C)e^{at}$ if $r=a$ is one of two roots of the auxiliary eqn
- $y_p(t)=t^2(At^m+Bt^{m-1}+\dots+C)e^{at}$ if $r=a$ is the repeated root of the auxiliary eqn.

3) $f(t)=(p_0t^{m_1}+p_1t^{m_1-1}+\dots+p_{m_1})\cos(bt)$
$+(q_0t^{m_2}+q_1t^{m_2-1}+\dots+q_{m_2})\sin(bt)$

$m=\max\{m_1,m_2\}$

Then

- $y_p(t)=(At^m+Bt^{m-1}+\dots+C)\cos(bt)$
$+(Dt^m+Et^{m-1}+\dots+F)\sin(bt)$
if $r=ib$ is not a root of the auxiliary eqn.
- $y_p(t)=t\left[(At^m+Bt^{m-1}+\dots+C)\cos(bt)\right.$
$\left.+(Dt^m+Et^{m-1}+\dots+F)\sin(bt)\right]$
if $r=ib$ is one of two roots of the auxiliary eqn.

4) $f(t)=\left[(p_0t^{m_1}+p_1t^{m_1-1}+\dots+p_{m_1})\cos(bt)\right.$
$\left.+(q_0t^{m_2}+q_1t^{m_2-1}+\dots+q_{m_2})\sin(bt)\right]e^{at}$

$m=\max\{m_1,m_2\}$

Then

- $y_p(t)=\left[(At^m+Bt^{m-1}+\dots+C)\cos(bt)\right.$
$\left.+(Dt^m+Et^{m-1}+\dots+F)\sin(bt)\right]e^{at}$

if $r=a+ib$ is not a root of the auxiliary eqn.

- $y_p(t)=\left[(At^m+Bt^{m-1}+\dots+C)\cos(bt)\right.$
$\left.+(Dt^m+Et^{m-1}+\dots+F)\sin(bt)\right]te^{at}$

if $r=a+ib$ is one of two roots of the auxiliary eqn.

1. Find the solution to the given initial value problem.

(a) $y'' + 4y = t^2 + 3e^t, \quad y(0) = 0, \quad y'(0) = 2.$

$y''+4y=t^2$

$y''+4y=3e^t$

General solution of the corresponding homogeneous eqn.

$y''+4y=0$

auxiliary eqn. is $r^2+4=0$

$r^2=-4$

$r=\pm 2i$

$y_h(t) = C_1\cos(2t)+C_2\sin(2t)$

Particular solutions of non homogeneous eqns

$y''+4y=\underbrace{t^2}_{\text{polynomial of degree 2}}$ (Case 1)

$r=0$ is <u>not</u> a root.

$y_{p_1}(t) = At^2+Bt+C$

$y''+4y=\underbrace{3}_{\text{constant = polynomial of degree 0}}e^{t}$

→ $r=1$ is not a root of the auxiliary eqn.

$y_{p_2}(t) = De^t$

$y_p = y_{p_1}+y_{p_2}$

$= At^2+Bt+C+De^t$

(b) $y'' + 4y = 3\sin 2t, \qquad y(0) = 2, \quad y'(0) = -1.$

$y'' + 4y = 0$

$r^2 + 4 = 0$

$r = \pm 2i$

$y_h(t) = C_1 \cos(2t) + C_2 \boxed{\sin(2t)}$ general solution of the homogeneous eqn.

$f(t) = 3\boxed{\sin(2t)}$

r=2i is a root of the auxiliary eqn.

$y_p(t) = [A\cos(2t) + B\sin(2t)]t$

(c) $y'' + 2y' + 5y = 4e^{-t}\cos 2t \qquad y(0) = 1, y'(0) = 0.$

Corresponding homogeneous eqn.

$y''+2y'+5y=0$

auxiliary eqn. $\quad r^2+2r+5=0$

$$r_1 = \frac{-2+\sqrt{4-20}}{2} = \frac{-2+\sqrt{-16}}{2} = \frac{-2+i\sqrt{16}}{2} = -1+2i$$

$$r_2 = \overline{r_1} = -1-2i$$

General solution of the homogeneous eqn. is

$$Re(r_1)=-1, \quad Im(r_1)=2$$

$$y_h(t) = e^{-t}(c_1\cos(2t)+c_2\sin(2t)) = c_1 \boxed{e^{-t}\cos(2t)} + c_2 e^{-t}\sin(2t)$$

$$f(t) = \underbrace{4}_{\text{constant}}\boxed{e^{-t}\cos(2t)}$$

$$y_p(t) = te^{-t}(A\cos(2t)+B\sin(2t))$$

$$y_p'(t) = (e^{-t}-te^{-t})(A\cos(2t)+B\sin(2t)) + te^{-t}(-2A\sin(2t)+2B\cos(2t))$$

$$\begin{aligned} y_p''(t) = {} & (-e^{-t}-e^{-t}+te^{-t})(A\cos(2t)+B\sin(2t)) \\ & + (e^{-t}-te^{-t})(-2A\sin(2t)+2B\cos(2t)) \\ & + (e^{-t}-te^{-t})(-2A\sin(2t)+2B\cos(2t)) \\ & + te^{-t}(-4A\cos(2t)-4B\sin(2t)) \end{aligned}$$

$$\begin{aligned} & (-2\cancel{e^{-t}} + \cancel{te^{-t}})(A\cos(2t)+B\sin(2t)) + 2(\cancel{e^{-t}}-\cancel{te^{-t}})(-2A\sin(2t)+2B\cos(2t)) \\ & + \cancel{te^{-t}}(-4A\cos(2t)-4B\sin(2t)) \\ & + \cancel{5te^{-t}}(A\cos 2t + 4B\sin(2t)) \\ & + 2(\cancel{e^{-t}}-\cancel{te^{-t}})(A\cos(2t)+B\sin(2t)) + \cancel{2te^{-t}}(-2A\sin(2t)+2B\cos(2t)) \\ & = 4\cancel{e^{-t}}\cos(2t) \end{aligned}$$

$$\cancel{-2(A\cos(2t)+B\sin(2t))} + 2(-2A\sin(2t)+2B\cos(2t)) \cancel{+2(A\cos(2t)+B\sin(2t))} = 4\cos(2t)$$

$$-4A\sin(2t)+4B\cos(2t) = 4\cos(2t)$$

$\sin(2t)$: $\quad -4A=0 \quad \Rightarrow \quad \boxed{A=0}$

$\cos(2t)$: $\quad 4B=4 \quad \Rightarrow \quad \boxed{B=1}$

$$y_p(t) = te^{-t}\sin(2t)$$

General solution:

$$\boxed{y(t) = e^{-t}(c_1\cos(2t)+c_2\sin(2t)) + te^{-t}\sin(2t)}$$

Plug into ICs.

2. Determine a suitable form for $y(t)$ if the method of undetermined coefficients is to be used

(a) $y'' + 3y' = 2t^4 + t^2e^{-3t} + \sin 3t$.

$y'' + 3y' = 0$

auxiliary eqn.

$r^2 + 3r = 0$

$r(r+3) = 0$

$r_1 = 0, \quad r_2 = -3$

$y_h(t) = C_1 + C_2 e^{-3t}$

$$f(t) = \underbrace{2t^4 e^{0\cdot t}}_{\substack{r=0 \\ \text{root}}} + \underbrace{t^2 e^{-3t}}_{\substack{r=-3 \\ \text{root}}} + \underbrace{\sin 3t}_{\substack{r=3i \\ \text{not a root}}}$$

$$y_p(t) = t\left(At^4 + Bt^3 + Ct^2 + Dt + E\right) + t\left(Ft^2 + Gt + H\right)e^{-3t} + I\sin(3t) + J\cos(3t)$$

(b) $y'' + 4y = t^2 \sin 2t + (6t+7)\cos 2t.$

$y'' + 4y = 0$

$r^2 + 4 = 0$

$r = \pm 2i$

$y_h(t) = C_1 \sin(2t) + C_2 \cos(2t)$

$f(t) = \underbrace{t^2}_{\text{polynomial of degree 2}} \sin(2t) + \underbrace{(6t+7)}_{\text{polynomial of degree 1}} \cos(2t)$

$2 = \max\{2, 1\}$

$y_p(t) = t\left[(At^2 + Bt + C)\sin(2t) + (Dt^2 + Et + F)\cos 2t\right]$

(c) $y'' + 2y' + 2y = 3e^{-t} + 2e^{-t}\cos t + 4t^2 e^{-t}\sin t$.

$$y'' + 2y' + 2y = 0$$

$$r^2 + 2r + 2 = 0$$

$$r_1 = \frac{-2 + \sqrt{4-8}}{2}$$

$$= -1 + i$$

$$Re(r_1) = -1, \quad Im(r_1) = 1$$

$$y_h(t) = e^{-t}(c_1 \cos t + c_2 \sin t)$$

$$f(t) = \underbrace{3e^{-t}}_{\substack{r=-1 \\ \text{not a root}}} + \underbrace{2e^{-t}\cos t + 4t^2 e^{-t}\sin t}_{\substack{r=-1+i \\ \text{a root}}}$$

$$\boxed{y_p(t) = Ae^{-t} + t\left[(Bt^2 + Ct + D)e^{-t}\cos t + (Et^2 + Ft + G)e^{-t}\sin t\right]}$$

3. Use the method of variation of parameters to find a particular solution to

(a) $y'' + y = \tan x$.

homogeneous eqn.

$$y'' + y = 0$$

auxiliary eqn:

$$r^2 + 1 = 0$$
$$r = \pm i$$

$$y_h(x) = C_1 \underbrace{\cos x}_{y_1(t)} + C_2 \underbrace{\sin x}_{y_2(x)}$$

$$\begin{cases} C_1'(x)\cos x + C_2'(x)\sin x = 0 \\ C_1'(x)(-\sin x) + C_2'(x)\cos x = \tan x \end{cases}$$

$$C_2'(x) = \frac{-C_1'(x)\cos x}{\sin x}$$

$$\left[C_1'(x)(-\sin x) - \frac{C_1'(x)\cos^2 x}{\sin x} = \tan x\right](\sin x)$$

$$C_1'(x)(-\sin^2 x) - C_1'(x)\cos^2 x = \frac{\sin^2 x}{\cos x}$$

$$C_1'(x)\underbrace{(-\sin^2 x - \cos^2 x)}_{-1} = \frac{\sin^2 x}{\cos x}$$

$$C_1'(x) = -\frac{\sin^2 x}{\cos x}$$

$$C_1(x) = -\int \frac{\sin^2 x}{\cos x}\,dx$$

$$= -\int \frac{1 - \cos^2 x}{\cos x}\,dx$$

$$= -\int (\sec x - \cos x)\,dx$$

$$\boxed{C_1(x) = -\ln|\sec x + \tan x| - \sin(x) + C_3}$$

$$C_2'(x) = -\frac{C_1'(x)\cos x}{\sin x}$$

$$= \sin x$$

$$\boxed{C_2(x) = -\cos x + C_4}$$

General solution of the nonhomogeneous eqn

$$\boxed{y(x) = [-\ln|\sec x + \tan x| - \sin x + C_3]\cos x + [-\cos x + C_4]\sin x}$$

Variation of parameters.

$$y'' + p(t)y' + q(t)y = g(t)$$

General solution of the corresponding homogeneous eqn:

$$y_h(t) = C_1 y_1(t) + C_2 y_2(t)$$

here $y_1(t)$ and $y_2(t)$ constitute the fundamental solution set.

Assume $C_1 = C_1(t)$, $C_2 = C_2(t)$ unknown functions

$$\begin{cases} C_1'(t)y_1(t) + C_2'(t)y_2(t) = 0 \\ C_1'(t)y_1'(t) + C_2'(t)y_2'(t) = g(t) \end{cases}$$

(b) $y'' + 4y' + 4y = t^{-2}e^{-2t}$.

$$= \frac{e^{-2t}}{t^2}$$

homogeneous eqn.

$$y'' + 4y' + 4y = 0$$

auxiliary eqn:

$$r^2 + 4r + 4 = 0$$

$$(r+2)^2 = 0$$

$r = -2$ repeated root

$$y_h(t) = (C_1 + C_2 t)e^{-2t}$$

$$= \underbrace{C_1 e^{-2t}}_{y_1(t)} + \underbrace{C_2 t e^{-2t}}_{y_2(t)}$$

$$\begin{cases} C_1' \cancel{e^{-2t}} + C_2' t \cancel{e^{-2t}} = 0 \\ C_1'(-2\cancel{e^{-2t}}) + C_2'(\cancel{e^{-2t}} - 2t\cancel{e^{-2t}}) = \frac{\cancel{e^{-2t}}}{t^2} \end{cases}$$

$$C_1' = -C_2' t$$

$$-2C_1' + C_2' - 2t\, C_2' = \frac{1}{t^2}$$

$$-2(-C_2' t) + C_2' - 2t\, C_2' = \frac{1}{t^2}$$

$$C_2' = \frac{1}{t^2}$$

$$\boxed{C_2(t) = -\frac{1}{t} + C_3}$$

$$C_1' = -C_2' t$$

$$= -\frac{1}{t}$$

$$\boxed{C_1(t) = -\ln|t| + C_4}$$

$$\boxed{y(t) = \left(-\frac{1}{t} + C_3\right)e^{-2t} + \left(-\ln|t| + C_4\right)te^{-2t}}$$

(c) $y'' - 2y' + y = \frac{e^t}{(1+t^2)}$.

4. A mass weighing 3 lb stretches a spring 3 in. If the mass is pushed upward, contracting the spring a distance of 1 in. then set in motion with a downward velocity of 2 ft/s, and if there is no damping, find the position u of the mass at any time t. Determine the frequency, period and amplitude of the motion.

$$\underbrace{m}_{\text{mass}}u'' + \underbrace{b}_{\text{damping const}}u' + \underbrace{k}_{\text{stiffness}}u = 0 \quad (\text{no external forces})$$

$$b = 0 \ (\text{no damping})$$

$W = 3\ lb$

$W = mg$

$m = \frac{W}{g} = \frac{3}{32}$

Hook's Law $\quad W = k\boxed{x}$ elongation

$$3 = k \frac{3}{12} \text{ ft}$$

$$k = 12$$

IVP:

$$\boxed{\begin{array}{l} \frac{3}{32}u'' + 12u = 0 \\ u(0) = -\frac{1}{12} \\ u'(0) = 2 \end{array}} \qquad \boxed{\begin{array}{l} u'' + 128u = 0 \\ u(0) = -\frac{1}{12} \\ u'(0) = 2 \end{array}}$$

5. A series circuit has a capacitor of 10^{-5} F, a resistor of $3 \times 10^2\ \Omega$, and an inductor of 0.2 H. The initial charge of the capacitor is 10^{-6} C and there is no initial current. Find the charge Q on the capacitor at any time t.

$L = 0.2$ $\qquad Q(0) = 10^{-6}$

$R = 300$ $\qquad Q'(0) = 0$

$C = 10^{-5}$

$$LQ'' + RQ' + \frac{1}{C} Q = 0.$$

$$0.2\,Q'' + 300\,Q' + 10^5 Q = 0$$

$$\boxed{\begin{gathered} Q'' + 1500\,Q' + 5 \times 10^5 Q = 0. \\ Q(0) = 10^{-6} \\ Q'(0) = 0. \end{gathered}}$$

6. A mass ~~of~~ weighting 4 lb stretches a spring 1.5 in. The mass is displaced 2 in. in the positive direction from its equilibrium position and released with no initial velocity. Assuming that there is no damping and that the mass is acted on by an external force of $2\cos 3t$ lb, formulate the initial value problem describing the motion of the mass.

$$W = 4$$

$$4 = mg$$

$$m = \frac{4}{g} = \frac{4}{32} = \frac{1}{8}$$

Hook's law: $W = k\,t$

$$4 = k \cdot \frac{1.5}{12}$$

$$k = \frac{48}{1.5} = 32$$

$$b = 0.$$

$$\frac{1}{8}u'' + 32u = 2\cos 3t$$

$$\boxed{\begin{aligned} u'' + 256u &= 16\cos 3t \\ u(0) &= \frac{2}{12} \\ u'(0) &= 0 \end{aligned}}$$