1. Find the Laplace transform of the given function using the definition:

   (a) $f(\cancel{x}\,t) = te^{3t}$.

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty f(t)e^{-st}\,dt$$ definition of the Laplace tranform.

$$\mathcal{L}\{te^{3t}\} = \int_0^\infty te^{3t}e^{-st}\,dt$$

$$= \int_0^\infty te^{(3-s)t}\,dt$$

integrate by parts

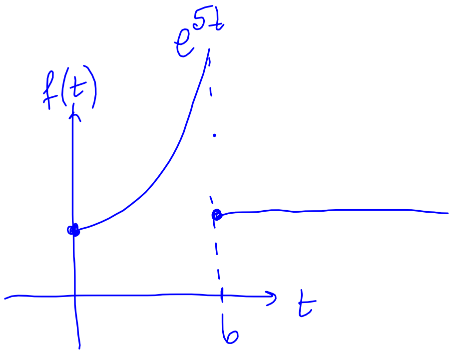
$$= \frac{1}{3-s}\left[\cancelto{0}{te^{(3-s)t}}\Big|_{t=0}^{t=\infty} - \int_0^\infty e^{(3-s)t}\,dt\right]$$

$$= -\frac{1}{3-s}\,\frac{1}{3-s}\,e^{(3-s)t}\Big|_{t=0}^{t=\infty}$$

$$\lim_{t\to\infty} e^{(3-s)t} = 0$$

$$= \boxed{\frac{1}{(3-s)^2}}$$

(b) $f(t) = \begin{cases} e^{5t} & 0 \le t < 6 \\ 3 & t \ge 6. \end{cases}$



$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$= \int_0^6 e^{5t} e^{-st}\, dt + \int_6^\infty 3e^{-st}\, dt$$

$$= \int_0^6 e^{(5-s)t}\, dt + 3 \cdot \frac{1}{-s} e^{-st} \Big]_{t=6}^{t=\infty}$$

$$= \frac{1}{5-s} e^{(5-s)t} \Big]_{t=0}^{t=6} + \frac{3}{-s}\left(0 - e^{-6s}\right)$$

$$= \boxed{\frac{1}{5-s} e^{6(5-s)} - \frac{1}{5-s} + \frac{3e^{-6s}}{s}}$$

Table of Laplace transform

$f(t) = \mathcal{L}^{-1}\{F\}(t)$	$F(s) = \mathcal{L}\{f\}(s)$
1	$\frac{1}{s}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
$t^n, \quad n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}, \quad s > 0$
$\sin bt$	$\frac{b}{s^2+b^2}, \quad s > 0$
$\cos bt$	$\frac{s}{s^2+b^2}, \quad s > 0$
$e^{at}t^n, \quad n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
$u(t-a)$	$\frac{e^{-as}}{s}$

Properties of Laplace transform

$$\mathcal{L}\{f+g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$$
$$\mathcal{L}\{cf\} = c\mathcal{L}\{f\} \quad \text{for any constant } c$$
$$\mathcal{L}\{e^{at}f\}(s) = F(s-a)$$
$$\boxed{\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)}$$
$$\boxed{\mathcal{L}\{f''\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)}$$
$$\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$$
$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n}(\mathcal{L}\{f(t)\})(s)$$
$$\mathcal{L}\{g(t)u(t-a)\}(s) = e^{-as}\mathcal{L}\{g(t+a)\}(s)$$
$$\mathcal{L}^{-1}\{e^{-as}\mathcal{L}\{f(t)\}(s)\} = f(t-a)u(t-a)$$

2. Find the inverse Laplace transform of

$\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}$

$\mathcal{L}\{e^{at}t^n\} = \frac{n!}{(s-a)^{n+1}}$

(a) $F(s) = \dfrac{3}{s^2+4} + \dfrac{5}{(s-1)^3}$.

$$\mathcal{L}^{-1}\left\{\frac{3}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{2}\cdot\frac{2}{s^2+2^2}\right\} = \frac{3}{2}\sin 2t$$

$$\frac{3}{s^2+4} = \frac{3}{s^2+2^2} = \frac{3}{2}\cdot\frac{2}{s^2+2^2}$$

$$\mathcal{L}^{-1}\left\{\frac{5}{(s-1)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{5}{2}\cdot\frac{2!}{(s-1)^3}\right\} = \frac{5}{2}t^2e^t$$

$$\frac{5}{(s-1)^3} = \frac{2!}{(s-1)^3}\cdot\frac{5}{2!}$$

$$\mathcal{L}^{-1}\left\{\frac{3}{s^2+4} + \frac{5}{(s-1)^3}\right\} = \frac{3}{2}\sin 2t + \frac{5}{2}t^2e^t$$

(b) $F(s) = \dfrac{2s^3 + 3s^2 - 8s + 12}{s^4 - 4s^2}$. $= \frac{2}{s} - \frac{3}{s^2} + \frac{3}{2} \cdot \frac{1}{s-2} - \frac{3}{2} \frac{1}{s+2}$

partial fraction decomposition

$$\frac{2s^3 + 3s^2 - 8s + 12}{s^4 - 4s^2} = \frac{2s^3 + 3s^2 - 8s + 12}{s^2(s^2-4)} = \frac{2s^3 + 3s^2 - 8s + 12}{s^2(s-2)(s+2)}$$

$$= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-2} + \frac{D}{s+2}$$

$A = 2$

$B = -3$

$C = \frac{3}{2}$

$D = -\frac{3}{2}$

$$\mathcal{L}^{-1}\left\{ \frac{2}{s} - \frac{3}{s^2} + \frac{3}{2} \cdot \frac{1}{s-2} - \frac{3}{2} \frac{1}{s+2} \right\}$$

$$= \boxed{2 - 3t + \frac{3}{2} e^{2t} - \frac{3}{2} e^{-2t}}$$

(c) $F(s) = \dfrac{2s+1}{s^2-2s+2}$. complete the square in the denominator

$$= \frac{2s+1}{(s-1)^2+1}$$

$$\mathcal{L}\{e^t \cos t\} = \frac{s-1}{(s-1)^2+1}$$

$$\mathcal{L}\{e^t \sin t\} = \frac{1}{(s-1)^2+1}$$

$$= \frac{\overbrace{2(s-1)}^{2s-2}+2+1}{(s-1)^2+1}$$

$$= \frac{2(s-1)+3}{(s-1)^2+1}$$

$$= \frac{2(s-1)}{(s-1)^2+1} + \frac{3}{(s-1)^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{2(s-1)}{(s-1)^2+1} + \frac{3}{(s-1)^2+1}\right\}$$

$$= 2e^t \cos t + 3e^t \sin t$$

3. Use the Laplace transform to solve the initial value problems:

(a) $y'' + 3y' + 2y = 0. \qquad y(0) = 1, \quad y'(0) = 0.$

$$\mathcal{L}\{y'' + 3y' + 2y\} = \mathcal{L}\{0\}$$

$$\mathcal{L}\{0\} = 0$$

$$\mathcal{L}\{y\} = Y(s)$$

$$\mathcal{L}\{y'\} = sY(s) - y(0)$$
$$= sY(s) - 1$$

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0)$$
$$= s^2Y(s) - s$$

$$\underbrace{s^2Y(s) - s}_{\mathcal{L}\{y''\}} + 3\underbrace{(sY(s) - 1)}_{\mathcal{L}\{y'\}} + 2\underbrace{Y(s)}_{\mathcal{L}\{y\}} = 0$$

$$Y(s)(s^2 + 3s + 2) - 3 - s = 0$$

solve for $Y(s)$

$$Y(s) = \frac{3+s}{s^2+3s+2}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{3+s}{s^2+3s+2}\right\}$$

$$\frac{3+s}{s^2+3s+2} = \frac{-1}{s+2} + \frac{2}{s+1}$$

$$y(t) = \mathcal{L}^{-1}\left\{-\frac{1}{s+2} + \frac{2}{s+1}\right\}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\boxed{y(t) = -e^{-2t} + 2e^{-t}}$$

(b) $y'' - 2y' + 2y = \cos t, \qquad y(0) = 1, \quad y'(0) = 0.$

(c) $y'' + 2y' + y = 4e^{-t} \qquad y(0) = 2, \quad y'(0) = -1.$