## Brief table of Laplace transform

| $f(t)$ | $F(s) = \mathcal{L}\{f\}(s)$ |
|---|---|
| $1$ | $\frac{1}{s}, \quad s > 0$ |
| $e^{at}$ | $\frac{1}{s-a}, \quad s > a$ |
| $t^n, \quad n = 1, 2, ...$ | $\frac{n!}{s^{n+1}}, \quad s > 0$ |
| $\sin bt$ | $\frac{b}{s^2 + b^2}, \quad s > 0$ |
| $\cos bt$ | $\frac{s}{s^2 + b^2}, \quad s > 0$ |
| $e^{at} t^n, \quad n = 1, 2, ...$ | $\frac{n!}{(s-a)^{n+1}}, \quad s > a$ |
| $e^{at} \sin bt$ | $\frac{b}{(s-a)^2 + b^2}, \quad s > a$ |
| $e^{at} \cos bt$ | $\frac{s-a}{(s-a)^2 + b^2}, \quad s > a$ |
| $u_c(t)$ | $\frac{e^{-cs}}{s}$ |
| $\delta(t - t_0)$ | $e^{-st_0}$ |

## Properties of Laplace transform

1. $\mathcal{L}\{f+g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$
2. $\mathcal{L}\{cf\} = c\mathcal{L}\{f\}$ for any constant $c$
3. $\mathcal{L}\{e^{at}f\}(s) = F(s-a)$
4. $\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$
5. $\mathcal{L}\{f''\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)$
6. $\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$
7. $\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \dfrac{d^n}{ds^n}(\mathcal{L}\{f(t)\})(s)$
8. $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s)$
9. $u_c(t)f(t-c) = \mathcal{L}^{-1}\{e^{-cs}F(s)\}$, where $f(t) = \mathcal{L}^{-1}\{F(s)\}$

**Convolution integrals**. If $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$ both exist for $s > a \geq 0$, then

$$H(s) = F(s)G(s) = \mathcal{L}\{h(t)\}, \quad s > a,$$

where

$$h(t) = \int_0^\tau f(t-\tau)g(\tau)d\tau = \int_0^\tau f(\tau)g(t-\tau)d\tau.$$

1. Express $f(t)$ in terms of the unit step function $u_c(t)$ and find its Laplace transform.

(a) $f(t) = \begin{cases} t^2, & 0 \le t < 2 \\ e^t, & 2 \le t \end{cases}$.

$$u_c(t) = \begin{cases} 0, & 0 \le t < c \\ 1, & t \ge c \end{cases}$$

$$f(t) = t^2 + u_2(t)(e^t - t^2)$$
$$= t^2 + u_2(t)e^t - t^2 u_2(t)$$

$$\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2\} + \mathcal{L}\{u_2(t)e^t\} - \mathcal{L}\{t^2 u_2(t)\}$$

$$\mathcal{L}\{t^2\} = \frac{2}{s^3}; \quad \mathcal{L}\{u_2(t)\} = \frac{e^{-2s}}{s}$$

property 7

$$\mathcal{L}\{t^2 u_2(t)\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{e^{-2s}}{s}\right)$$
$$= \left(\frac{-2e^{-2s}s - e^{-2s}}{s^2}\right)' = \left(-\frac{2e^{-2s}}{s} - \frac{e^{-2s}}{s^2}\right)'$$
$$= -\frac{-4e^{-2s}s - 2e^{-2s}}{s^2} - \frac{-2e^{-2s}(s^2) - 2s\,e^{-2s}}{s^4}$$
$$= \boxed{\frac{4e^{-2s}}{s} + \frac{4e^{-2s}}{s^2} + \frac{2e^{-2s}}{s^3} = \mathcal{L}\{t^2 u_2(t)\}}$$

$$\mathcal{L}\{u_2(t)e^t\} = \mathcal{L}\{u_2(t)e^{t-2}\cdot e^2\}$$
$$= e^2\mathcal{L}\{u_2(t)e^{t-2}\}$$
$$= e^2 e^{-2s}\mathcal{L}\{e^t\} \qquad \mathcal{L}\{e^t\} = \frac{1}{s-1}$$
$$= \frac{e^{2-2s}}{s-1}$$

$$\boxed{\mathcal{L}\{f(t)\} = \frac{2}{s^3} + \frac{e^{2-2s}}{s-1} - \frac{4e^{-2s}}{s} - \frac{4e^{-2s}}{s^2} - \frac{2e^{-2s}}{s^3}}$$

(b) $f(t) = \begin{cases} 2, & 0 \le t < 3 \\ 5t^2, & 3 \le t < 8 \\ 3\cos(t-8), & 8 \le t \end{cases}$.

$$f(t) = 2 + u_3(t)(5t^2 - 2) + u_8(t)(3\cos(t-8) - 5t^2)$$

$$= 2 + 5t^2(u_3(t) - u_8(t)) - 2u_3(t) + 3\cos(t-8)u_8(t)$$

$$\mathcal{L}\{f(t)\} = 2\mathcal{L}\{1\} + 5\mathcal{L}\{t^2(u_3(t) - u_8(t))\} - 2\mathcal{L}\{u_3(t)\} + 3\mathcal{L}\{\cos(t-8)u_8(t)\}$$

$$\mathcal{L}\{1\} = \frac{1}{s}$$

$$\mathcal{L}\{t^2(u_3(t) - u_8(t))\} \underset{\text{property 7}}{=} (-1)^2 \frac{d^2}{ds^2}\left(\mathcal{L}(u_3(t) - u_8(t))\right)$$

$$\mathcal{L}\{u_3(t) - u_8(t)\} = \mathcal{L}\{u_3(t)\} - \mathcal{L}\{u_8(t)\}$$

$\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}$

$$= \frac{e^{-3s}}{s} - \frac{e^{-8s}}{s}$$

$$\mathcal{L}\{t^2(u_3(t) - u_8(t))\} = \left(\frac{e^{-3s}}{s} - \frac{e^{-8s}}{s}\right)''$$

$$= \left(\frac{-3e^{-3s}s - e^{-3s}}{s^2} - \frac{-8e^{-8s}s - e^{-8s}}{s^2}\right)'$$

$$= \frac{(9e^{-3s}s - 3e^{-3s} + 3e^{-3s})s^2 - 2s(-3e^{-3s}s - e^{-3s})}{s^4} - \frac{(64e^{-8s}s - 8e^{-8s} + 8e^{-8s})s^2 - 2s(-8e^{-8s}s - e^{-8s})}{s^4}$$

$$= \frac{9e^{-3s}}{s} + \frac{6e^{-3s}}{s^2} + \frac{2e^{-3s}}{s^3} - \frac{64e^{-8s}}{s} - \frac{16e^{-8s}}{s^2} - \frac{2e^{-8s}}{s^3}$$

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$$\mathcal{L}\{u_3(t)\} = \frac{e^{-3s}}{s}$$

$$\mathcal{L}\{\cos(t-8)u_8(t)\} = e^{-8s}\underbrace{\mathcal{L}\{\cos t\}}_{\frac{s}{s^2+1}} = \frac{se^{-8s}}{s^2+1}$$

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$$\mathcal{L}\{f(t)\} = \frac{2}{s} + 5\left(\frac{9e^{-3s}}{s} + \frac{6e^{-3s}}{s^2} + \frac{2e^{-3s}}{s^3} - \frac{64e^{-8s}}{s} - \frac{16e^{-8s}}{s^2} - \frac{2e^{-8s}}{s^3}\right) - \frac{2e^{-3s}}{s} + \frac{3se^{-8s}}{s^2+1}$$

2. Find the inverse Laplace transform of the given functions

(a) $F(s) = \dfrac{s + 3se^{-5s}}{s^2 - 4s + 3} = \dfrac{s}{s^2-4s+3} + 3\dfrac{s}{s^2-4s+3}e^{-5s}$

$$\frac{s}{s^2-4s+3} = \frac{s}{(s-3)(s-1)} = \frac{A}{s-3} + \frac{B}{s-1}$$

$$\frac{s}{(s-3)(s-1)} = \frac{A(s-1)+B(s-3)}{(s-3)(s-1)}$$

$$s = A(s-1) + B(s-3)$$

$s=1$: $1 = -2B$  $\boxed{B = -1/2}$

$s=3$: $3 = 2A$  $\boxed{A = 3/2}$

$$\frac{s}{s^2-4s+3} = \frac{3}{2}\frac{1}{s-3} - \frac{1}{2}\frac{1}{s-1}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2-4s+3}\right\} = \frac{3}{2}\underbrace{\mathcal{L}\left\{\frac{1}{s-3}\right\}}_{e^{3t}} - \frac{1}{2}\underbrace{\mathcal{L}\left\{\frac{1}{s-1}\right\}}_{e^{t}}$$

$$= \frac{3}{2}e^{3t} - \frac{1}{2}e^{t}$$

$$\mathcal{L}^{-1}\left\{\frac{se^{-5s}}{s^2-4s+3}\right\} = u_5(t)\left[\frac{3}{2}e^{3(t-5)} - \frac{1}{2}e^{t-5}\right]$$

$\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)f(t-c)$, where $F(s) = \mathcal{L}\{f(t)\}$

$$\boxed{\mathcal{L}^{-1}\{F(s)\} = \frac{3}{2}e^{3t} - \frac{1}{2}e^{t} + 3u_5(t)\left[\frac{3}{2}e^{3(t-5)} - \frac{1}{2}e^{t-5}\right]}$$

(b) $F(s) = \dfrac{(2s-1)e^{-s}}{s^2 - 2s + 2}$.

$\mathcal{L}\{e^{at}\cos bt\} = \dfrac{s-a}{(s-a)^2+b^2}$

$\mathcal{L}\{e^{at}\sin bt\} = \dfrac{b}{(s-a)^2+b^2}$

$$s^2-2s+2 = s^2-2s+1+1 = (s-1)^2+1$$

$$\mathcal{L}^{-1}\left\{\frac{2s-1}{s^2-2s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{2s-1}{(s-1)^2+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2s}{(s-1)^2+1}\right\} - \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+1}\right\}}_{e^t \sin t}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{(s-1)+1}{(s-1)^2+1}\right\} - e^t \sin t$$

$$= 2\underbrace{\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+1}\right\}}_{e^t\cos t} + 2\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+1}\right\}}_{e^t \sin t} - e^t \sin t$$

$$= 2e^t\cos t + 2e^t \sin t - e^t \sin t$$

$$= 2e^t \cos t + e^t \sin t$$

$$\boxed{\mathcal{L}^{-1}\left\{\frac{2s-1}{s^2-2s+2}e^{-s}\right\} = u_1(t)\left(2e^{t-1}\cos(t-1) + e^{t-1}\sin(t-1)\right)}$$

3. Find the solution to the given initial value problem

(a) $y'' + 3y' + 2y = \begin{cases} 1, & 0 \le t < 10 \\ 0, & 10 \le t \end{cases}, \qquad y(0) = 0, \quad y'(0) = 0.$

$$g(t) = \begin{cases} 1, & 0 \le t < 10 \\ 0, & t \ge 10 \end{cases}$$

$$g(t) = 1 + (0-1)u_{10}(t) = 1 - u_{10}(t)$$

$$y'' + 3y' + 2y = 1 - u_{10}(t)$$

$$\mathcal{L}\{y'' + 3y' + 2y\} = \mathcal{L}\{1 - u_{10}(t)\}$$

$$\mathcal{L}\{y\} = Y(s)$$

$$\mathcal{L}\{y'\} = sY(s) - y(0) = sY(s)$$

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s)$$

$$\mathcal{L}\{1 - u_{10}(t)\} = \frac{1}{s} - \frac{e^{-10s}}{s}$$

$$s^2Y(s) + 3sY(s) + 2Y(s) = \frac{1}{s} - \frac{e^{-10s}}{s}$$

$$Y(s)(s^2+3s+2) = \frac{1}{s} - \frac{e^{-10s}}{s}$$

$$Y(s) = \frac{1}{s(s^2+3s+2)} - \frac{e^{-10s}}{s(s^2+3s+2)}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+3s+2)} - \frac{e^{-10s}}{s(s^2+3s+2)}\right\}$$

$$\frac{1}{s(s^2+3s+2)} = \frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$\frac{1}{s(s+1)(s+2)} = \frac{A(s+1)(s+2) + Bs(s+2) + Cs(s+1)}{s(s+1)(s+2)}$$

$s=0$: $1 = 2A$ $\quad \boxed{A = 1/2}$

$s=-1$: $1 = -B$ $\quad \boxed{B = -1}$

$s=-2$: $1 = 2C$ $\quad \boxed{C = 1/2}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+3s+2)}\right\} = \frac{1}{2}\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}}_{1} - \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}}_{e^{-t}} + \frac{1}{2}\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}}_{e^{-2t}}$$

$$= \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-10s}}{s(s^2+3s+2)}\right\} = u_{10}(t)\left[\frac{1}{2} - e^{-(t-10)} + \frac{1}{2}e^{-2(t-10)}\right]$$

$$\boxed{y(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} - u_{10}(t)\left[\frac{1}{2} - e^{-(t-10)} + \frac{1}{2}e^{-2(t-10)}\right]}$$

(b) $y'' + 2y' + 5y = \sin(t) + u_\pi(t)\cos(t-\pi), \qquad y(0) = 0, \quad y'(0) = 0.$

$\mathcal{L}\{y''+2y'+5y\} = \mathcal{L}\{\sin t + u_\pi(t)\cos(t-\pi)\}$

$\mathcal{L}\{y\} = Y(s)$

$\mathcal{L}\{y'\} = sY(s) - y(0) = sY(s)$

$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s)$

$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$

$\mathcal{L}\{u_\pi(t)\cos(t-\pi)\} = e^{-\pi s}\mathcal{L}\{\cos t\} = \frac{e^{-\pi s}s}{s^2+1}$

$Y(s)(s^2+2s+5) = \frac{1}{s^2+1} + \frac{se^{-\pi s}}{s^2+1}$

$Y(s) = \frac{1}{(s^2+1)(s^2+2s+5)} + \frac{se^{-\pi s}}{(s^2+1)(s^2+2s+5)}$

$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+2s+5)}\right\} + \mathcal{L}^{-1}\left\{\frac{se^{-\pi s}}{(s^2+1)(s^2+2s+5)}\right\}$

$\frac{1}{(s^2+1)(s^2+2s+5)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+5}$

$\frac{1}{(s^2+1)(s^2+2s+5)} = \frac{(As+B)(s^2+2s+5)+(Cs+D)(s^2+1)}{(s^2+1)(s^2+2s+5)}$

$s=0: \quad 1 = 5B + D$

$s=1: \quad 1 = (A+B)8 + (C+D)2$

$s=-1: \quad 1 = (-A+B)4 + (-C+D)2$

$s=2: \quad 1 = (2A+B)13 + (2C+D)5$

$D = 1-5B$

$2 = 4A + 12B + 4D$

$1 = 2A + 6B + 2D$

$1 = 2A + 6B + 2 - 10B$

$-1 = 2A - 4B$

$1 = 8A + 8B + 2C + 2D$

$-\;1 = -4A + 4B - 2C + 2D$

$0 = 12A + 4B + 4C$

$0 = 3A + B + C$

$1 = 26A + 13B + 10C + 5 - 25B$

$-4 = 26A - 12B + 10C$

$-2 = 13A - 6B + 5C$

$\begin{cases} -1 = 2A - 4B \\ 0 = 3A + B + C \\ -2 = 13A - 6B + 5C \end{cases} \qquad A = \frac{4B-1}{2}$

$\begin{cases} 0 = \frac{3}{2}(4B-1) + B + C \quad |\,2 \\ -2 = \frac{13}{2}(4B-1) - 6B + 5C \quad |\,2 \end{cases}$

$0 = 3(4B-1) + 2B + 2C$

$\begin{cases} 0 = 14B - 3 + 2C \\ -4 = 40B - 13 + 10C \end{cases} \qquad \begin{cases} 14B + 2C = 3 \\ 40B + 10C = 9 \end{cases}$

$2C = 3 - 14B$

$40B + 5(3-14B) = 9$

$15 - 30B = 9$

$B = \frac{6}{30} = \frac{1}{5}$

$C = \frac{1}{10}$

$A = \frac{4B-1}{2} = -\frac{1}{10}$

$D = 1 - 5B = 0$

$$\frac{1}{(s^2+1)(s^2+2s+5)} = \frac{-\frac{1}{10}s+\frac{1}{5}}{s^2+1} + \frac{\frac{1}{10}s}{s^2+2s+5}$$

$$= -\frac{1}{10}\frac{s}{s^2+1} + \frac{1}{5}\frac{1}{s^2+1} + \frac{1}{10}\frac{s}{s^2+2s+5}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+2s+5)}\right\} = -\frac{1}{10}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$+\frac{1}{10}\mathcal{L}^{-1}\left\{\frac{(s+1)-1}{(s+1)^2+4}\right\} = -\frac{1}{10}\cos t + \frac{1}{5}\sin t$$

$$+\frac{1}{10}\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+4}\right\} - \frac{1}{10}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+4}\right\}$$

$$= -\frac{1}{10}\cos t + \frac{1}{5}\sin t + \frac{1}{10}e^{-t}\cos 2t - \frac{1}{10}e^{-t}\sin 2t$$

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$$\mathcal{L}^{-1}\left\{\frac{s\, e^{-\pi s}}{(s^2+1)(s^2+2s+5)}\right\}$$

$$\frac{s}{(s^2+1)(s^2+2s+5)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+5}$$

$$= \frac{(As+B)(s^2+2s+5)+(Cs+D)(s^2+1)}{(s^2+1)(s^2+2s+5)}$$

$$\frac{s}{(s^2+1)(s^2+2s+5)} = \frac{s^3(A+C)+s^2(B+2A+D)+s(5A+2B+C)+(5B+D)}{(s^2+1)(s^2+2s+5)}$$

$s^3$: $0 = A+C$

$s^2$: $B+2A+D = 0$

$s$: $5A+2B+C = 1$

$1$: $5B+D = 0$

$A = -C$

$D = -5B$

$-4B+2A = 0$

$A = 2B$  $C = -2B$

$10B+2B-2B = 1$

$B = \frac{1}{10}$

$A = \frac{1}{5}$, $C = -\frac{1}{5}$

$D = -\frac{1}{2}$

finish it.

(c) $y'' + 2y' + 2y = \cos t + \delta(t - \pi/2), \qquad y(0) = 0, \quad y'(0) = 0.$

$$\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}$$

$$\mathcal{L}\{y'' + 2y' + 2y\} = \mathcal{L}\{\cos t + \delta(t - \pi/2)\}$$

$$\mathcal{L}\{y\} = Y(s)$$

$$(s^2 + 2s + 2)Y(s) = \frac{s}{s^2+1} + e^{-\frac{\pi}{2}s}$$

$$Y(s) = \frac{s}{(s^2+1)(s^2+2s+2)} + \frac{e^{-\frac{\pi}{2}s}}{s^2+2s+2}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)(s^2+2s+2)} + \frac{e^{-\frac{\pi}{2}s}}{s^2+2s+2}\right\}$$

partial fractions

$$\mathcal{L}^{-1}\left\{\frac{e^{-\pi/2\, s}}{s^2+2s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\, e^{-\frac{\pi}{2}s}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} = e^{-t}\sin t$$

$$u_{\frac{\pi}{2}}(t)\, e^{-(t-\frac{\pi}{2})} \sin(t - \pi/2)$$

$$\boxed{y(t) = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)(s^2+2s+2)}\right\} + u_{\frac{\pi}{2}}(t)\, e^{-(t-\frac{\pi}{2})} \sin\left(t - \frac{\pi}{2}\right)}$$

(d) $y'' - y' - 6y = g(t), \qquad y(0) = 1, \quad y'(0) = 8.$

convolution theorem

$\mathcal{L}\{y\} = Y(s)$

$\mathcal{L}\{y'\} = sY(s) - 1$

$\mathcal{L}\{y''\} = s^2 Y(s) - s - 8$

$\mathcal{L}\{g(t)\} = G(s)$

$Y(s)(s^2 - s - 6) - s - 7 = G(s)$

$$Y(s) = \frac{G(s)}{s^2-s-6} + \frac{s+7}{s^2-s-6}$$

$$\frac{1}{s^2-s-6} = \frac{1}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2} = F(s)$$

$$\mathcal{L}^{-1}\left\{\frac{A}{s-3} + \frac{B}{s+2}\right\} = Ae^{3t} + Be^{-2t} = f(t)$$

regular product

$$\mathcal{L}^{-1}\left\{\frac{G(s)}{s^2-s-6}\right\} = \mathcal{L}^{-1}\left\{G(s)\cdot F(s)\right\}$$

convolution

$$= (f * g)(t)$$

$$= \int_0^t f(\tau)g(t-\tau)\,d\tau$$

$$= \int_0^t \left(Ae^{3\tau} + Be^{-2\tau}\right) g(t-\tau)\,d\tau$$

$$\mathcal{L}^{-1}\left\{\frac{s+7}{s^2-s-6}\right\} = \mathcal{L}^{-1}\left\{\frac{C}{s-3} + \frac{D}{s+2}\right\}$$

$$= Ce^{3t} + De^{-2t}$$

$$\boxed{y(t) = \int_0^t \left(Ae^{3\tau} + Be^{-2\tau}\right) g(t-\tau)\,d\tau + Ce^{3t} + De^{-2t}}$$

4. Find the inverse Laplace transform of the given function by using the convolution theorem.

(a) $F(s) = \dfrac{(3s-3)e^{-5s}}{s^2+2s+10}$.

(b) $F(s) = \dfrac{1}{s^4(s^2-1)}$.

(c) $F(s) = \dfrac{s}{(s+1)^2(s+4)^3}$.

5. Find the Laplace transform of

(a) $f(t) = \int_0^1 (t - \tau) e^{3\tau}\, d\tau$

(b) $f(t) = \int_0^1 e^{\tau} \sin(t - \tau)\, d\tau$

6. Transform the given equation into a system of first order equation, then in matrix notation.

   (a) $e^t y'' + t^2 y' - \sin t y = 3 \arctan t, \qquad y(0) = 5, \quad y'(0) = 3.$

(b) $y^{(4)} - \cos ty = 0.$

7. If $A = \begin{pmatrix} 1 & -2 & 0 \\ 3 & 2 & -1 \\ -2 & 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & -2 & 3 \\ -1 & 5 & 0 \\ 6 & 1 & 2 \end{pmatrix}$, find

(a) $3A - 2B$

(b) $AB - BA$

8. Verify that the vector $X(t) = \begin{pmatrix} 6 \\ -8 \\ -4 \end{pmatrix} e^{-t} + 2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t}$ is solution to the system

$$X' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

9. Verify that $\psi = \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix}$ is solution to

$$\psi' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \psi$$