1. Transform the given equation into a system of first order equation, then in matrix notation.

$y'' + e^{-t}t^2 y' - e^{-t}\sin t\, y = 3e^{-t}\arctan t$

(a) $e^t y'' + t^2 y' - \sin t y = 3\arctan t, \qquad y(0) = 5, \quad y'(0) = 3.$

$y = x_1$

$y' = x_2$

$y'' = x_2'$

$$\begin{cases} x_1' = x_2 \\ e^t x_2' + t^2 x_2 - x_1 \sin t = 3\arctan t \end{cases}$$

system

solve for x_2'

ICs: $y(0) = x_1(0) = 5, \quad x_2(0) = y'(0) = 3.$

$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 0 \\ 3e^{-t}\arctan t \end{pmatrix}, \quad A = \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{pmatrix} 0 & 1 \\ \sin t e^{-t} & -t^2 e^{-t} \end{pmatrix} \end{array}$

coefficient matrix

$x_2' = e^{-t}(-t^2 x_2 + x_1 \sin t) + (3\arctan t)e^{-t}$

matrix form:

$$\boxed{\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \sin t e^{-t} & -t^2 e^{-t} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 3e^{-t}\arctan t \end{pmatrix}}$$

$\vec{x}(0) = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$ ICs.

(b) $y'' - \cos t y' + 3t y = 0.$

$y'' = \cos t\, y' - 3t y$

$y = x_1$

$y' = x_2$

$y'' = x_2'$

$$\boxed{\begin{cases} x_1' = x_2 \\ x_2' = \cos t\, x_2 - 3t\, x_1 \end{cases}}$$

matrix form: $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, coefficient matrix $A = \begin{pmatrix} 0 & 1 \\ -3t & \cos t \end{pmatrix}$

$$\boxed{\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -3t & \cos t \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}$$

2. Transform the given system into a single equation of second order. Find x_1 and x_2 that satisfies the initial conditions when initial conditions are given:

(a) $\begin{cases} x_1' = x_1 - 2x_2, & x_1(0) = -1 \\ x_2' = 3x_1 - 4x_2, & x_2(0) = 2 \end{cases}$.

Solve 1st equation for x_2: $x_2 = \dfrac{x_1 - x_1'}{2}$

Plug x_1 into the 2nd equation: $x_2' = \dfrac{x_1' - x_1''}{2}$

$$\left[\frac{x_1' - x_1''}{2} = 3x_1 - 4\cdot\frac{x_1 - x_1'}{2}\right](2)$$

$$x_1' - x_1'' = 6x_1 - 4x_1 + 4x_1'$$

$$\boxed{\begin{aligned} x_1'' + 3x_1' + 2x_1 = 0 \\ x_1(0) = -1 \end{aligned}}$$

need the value for $x_1'(0)$ - ?

1st equation $x_1' = x_1 - 2x_2$

plug $t=0$: $x_1'(0) = x_1(0) - 2x_2(0)$

$= -1 - 2(2)$

$= -5$

$$\boxed{x_1'(0) = -5}$$

auxiliary eqn: $r^2 + 3r + 2 = 0$

$(r+2)(r+1) = 0$

$r_1 = -2,\ r_2 = -1$

$$x_1(t) = C_1 e^{-2t} + C_2 e^{-t}$$

Plug x_1 into ICs:

$x_1(0) = C_1 + C_2 = -1$

$+$ $x_1'(0) = -2C_1 - C_2 = -5$

$-C_1 = -6 \Rightarrow \boxed{C_1 = 6}$

$C_2 = -2C_1 + 5$

$= -12 + 5$

$= \boxed{-7 = C_2}$

$$\boxed{x_1(t) = 6e^{-2t} - 7e^{-t}}$$

$$x_2(t) = \frac{x_1 - x_1'}{2} = \frac{6e^{-2t} - 7e^{-t} - (-12e^{-2t} + 7e^{-t})}{2}$$

$$= \frac{18e^{-2t} - 14e^{-t}}{2}$$

$$\boxed{x_2(t) = 9e^{-2t} - 7e^{-t}}$$

(b) $\begin{cases} 2(x_1' + 2x_2' = 4x_1 + 5x_2) \\ 2x_1' - x_2' = 3x_1 \end{cases}$

$$\begin{cases} 2x_1' + 4x_2' = 8x_1 + 10x_2 \\ 2x_1' - x_2' = 3x_1 \end{cases}$$

$$\frac{5x_2' = 5x_1 + 10x_2}{5}$$

$$x_2' = x_1 + 2x_2$$

$$x_1 = x_2' - 2x_2$$

Plug x_1 into the 2nd equation:

$$2\underbrace{(x_2'' - 2x_2')}_{x_1'} - x_2' = 3\underbrace{(x_2' - 2x_2)}_{x_1}$$

$$2x_2'' - 5x_2' = 3x_2' - 6x_2$$

$$2x_2'' - 8x_2' + 6x_2 = 0$$

$$x_2'' - 4x_2' + 3x_2 = 0$$

auxiliary eqn: $r^2 - 4r + 3 = 0$

$(r-3)(r-1) = 0$

$r_1 = 3,\ r_2 = 1$

$$\boxed{x_2(t) = c_1 e^{3t} + c_2 e^t}$$

$$x_1(t) = x_2'(t) - 2x_2(t)$$

$$= 3c_1 e^{3t} + c_2 e^t - 2c_1 e^{3t} - 2c_2 e^t$$

$$= \boxed{c_1 e^{3t} - c_2 e^t = x_1(t)}$$

3. Verify that $\psi = \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix}$ is solution to

$$\psi' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \psi$$ matrix equation.

if $A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) \dots & a_{2n}(t) \\ \dots & \dots & \dots \\ a_{m1}(t) & a_{m2}(t) \dots & a_{mn}(t) \end{pmatrix}$, then

$$A'(t) = \begin{pmatrix} a_{11}'(t) & a_{12}'(t) & \dots & a_{1n}'(t) \\ a_{21}'(t) & a_{22}'(t) & \dots & a_{2n}'(t) \\ \dots & \dots & & \dots \\ a_{m1}'(t) & a_{m2}'(t) & \dots & a_{mn}'(t) \end{pmatrix}$$

$$\psi'(t) = \begin{pmatrix} -3e^{-3t} & 2e^{2t} \\ 12e^{-3t} & 2e^{2t} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \psi = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-3t} - 4e^{-3t} & e^{2t} + e^{2t} \\ 4e^{-3t} - 2(-4e^{-3t}) & 4e^{2t} - 2e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} -3e^{-3t} & 2e^{2t} \\ 12e^{-3t} & 2e^{2t} \end{pmatrix} = \psi'(t)$$

4. Given the matrices

$$A = \begin{pmatrix} -2 & 1 & 2 \\ 1 & 0 & -3 \\ 2 & -1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -1 & -1 \\ -2 & 1 & 0 \end{pmatrix}$$

(a) Find $3A - 2B$ $= 3\begin{pmatrix} -2 & 1 & 2 \\ 1 & 0 & -3 \\ 2 & -1 & 1 \end{pmatrix} - 2\begin{pmatrix} 1 & 2 & 3 \\ 3 & -1 & -1 \\ -2 & 1 & 0 \end{pmatrix}$

$$= \begin{pmatrix} -6 & 3 & 6 \\ 3 & 0 & -9 \\ 6 & -3 & 3 \end{pmatrix} - \begin{pmatrix} 2 & 4 & 6 \\ 6 & -2 & -2 \\ -4 & 2 & 0 \end{pmatrix} = \begin{pmatrix} -6-2 & 3-4 & 6-6 \\ 3-6 & 0-(-2) & -9-(-2) \\ 6-(-4) & -3-2 & 3-0 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} -8 & -1 & 0 \\ -3 & 2 & -7 \\ 10 & -5 & 3 \end{pmatrix}}$$

(b) Calculate $\det(A)$.

$$\begin{array}{cccccc} + & + & + & - & - & - \end{array}$$

$$\det A = \left| \begin{array}{ccc|ccc} -2 & 1 & 2 & -2 & 1 & 2 \\ 1 & 0 & -3 & 1 & 0 & -3 \\ 2 & -1 & 1 & 2 & -1 & 1 \end{array} \right| = 0 - 6 - 2 - 0 + 6 - 1$$

$$= \boxed{-3}$$

(c) Find $A^2 - AB$. $= A(A-B)$

$$A = \begin{pmatrix} -2 & 1 & 2 \\ 1 & 0 & -3 \\ 2 & -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -1 & -1 \\ -2 & 1 & 0 \end{pmatrix}$$

$$A^2 = AA = \begin{pmatrix} -2 & 1 & 2 \\ 1 & 0 & -3 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} -2 & 1 & 2 \\ 1 & 0 & -3 \\ 2 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2(-2)+1(1)+2(2) & -2(1)+1(0)+2(-1) & -2(2)+1(-3)+2(1) \\ 1(-2)+1(0)+2(-3) & 1(1)+0+(-3)(-1) & 1(2)+0(-3)+(-3)1 \\ 2(-2)-1(1)+1(2) & 2(1)-1(0)+1(-1) & 2(2)-1(-3)+1(1) \end{pmatrix}$$

$$= \begin{pmatrix} 9 & -4 & -5 \\ -8 & 4 & -1 \\ -3 & 1 & 8 \end{pmatrix}$$

$$AB = \begin{pmatrix} -2 & 1 & 2 \\ 1 & 0 & -3 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & -1 & -1 \\ -2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -2+3-4 & -4-1+2 & -6-1 \\ 1+6 & 2-3 & 3 \\ 2-3-2 & 4+1+1 & 6+1 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & -3 & -7 \\ 7 & -1 & 3 \\ -3 & 6 & 7 \end{pmatrix}$$

$$A^2 - AB = \begin{pmatrix} 9 & -4 & -5 \\ -8 & 4 & -1 \\ -3 & 1 & 8 \end{pmatrix} - \begin{pmatrix} -3 & -3 & -7 \\ 7 & -1 & 3 \\ -3 & 6 & 7 \end{pmatrix} = \boxed{\begin{pmatrix} 12 & -1 & 2 \\ -15 & 5 & -4 \\ 0 & -5 & 1 \end{pmatrix}}$$

5. Given the matrices and vectors

$$A = \begin{pmatrix} -2 & 0 & 1 \\ 1 & -1 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 3 \\ -1 & 0 \\ 3 & -1 \end{pmatrix}, \qquad X(t) = \begin{pmatrix} 2t \\ e^{-t} \end{pmatrix}, \qquad Y(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}.$$

(a) Find AB and BA

$$AB = \begin{pmatrix} -2 & 0 & 1 \\ 1 & -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ -1 & 0 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} -2+0+3(1) & -2(3)+1(-1) \\ 1+(-1)(-1)+3(3) & 1(3)-1(3) \end{pmatrix} = \begin{pmatrix} 1 & -7 \\ 11 & 0 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & 3 \\ -1 & 0 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} -2 & 0 & 1 \\ 1 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 1(-2)+3(1) & 1(0)+3(-1) & 1(1)+3(3) \\ -1(-2)+0(1) & (-1)(0)+\cancel{0}(-1) & (-1)1+3(0) \\ 3(-2)+(-1)(1) & 3(0)-(1)(-1) & 3(1)-1(3) \end{pmatrix}$$

$$= \begin{pmatrix} +1 & -3 & 10 \\ 2 & 0 & -1 \\ -7 & 1 & 0 \end{pmatrix}$$

(b) Find AX, BX, AY, BY if possible.

$$A = \begin{pmatrix} -2 & 0 & 1 \\ 1 & -1 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 3 \\ -1 & 0 \\ 3 & -1 \end{pmatrix}$$

$$x = \begin{pmatrix} 2t \\ e^{-t} \end{pmatrix}, \quad y(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}$$

$$Ax = \begin{pmatrix} -2 & 0 & 1 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} 2t \\ e^{-t} \end{pmatrix} \quad \text{not possible}$$

$$Bx = \begin{pmatrix} 1 & 3 \\ -1 & 0 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 2t \\ e^{-t} \end{pmatrix} = \begin{pmatrix} 1(2t) + 3e^{-t} \\ -1(2t) + 0e^{-t} \\ 3(2t) - e^{-t} \end{pmatrix} = \begin{pmatrix} 2t + 3e^{-t} \\ -2t \\ 6t - e^{-t} \end{pmatrix}$$

$$AY = \begin{pmatrix} -2 & 0 & 1 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix} = \begin{pmatrix} -2\cos t + 0\sin t + t \\ \cos t - \sin t + 3t \end{pmatrix} = \begin{pmatrix} -2\cos t + t \\ \cos t - \sin t + 3t \end{pmatrix}$$

$$BY = \begin{pmatrix} 1 & 3 \\ -1 & 0 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix} \quad \text{not possible}$$

$\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ are linearly independent if

$c_1\vec{x}_1 + c_2\vec{x}_2 + \ldots + c_n\vec{x}_n = \vec{0}$ if and only if $c_1 = c_2 = \ldots = c_n = 0$

if $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ are not linearly independent, then they are linearly dependent (one of the vectors is the linear combination of the other vectors)

6. Determine whether the vectors are linearly independent. If they are linearly dependent, find a linear relation among them:

(a) $X_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}.$

if we have n vectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ with n components then they are linearly dependent if

$\det(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n) = 0.$

if $\det(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n) \neq 0$, then $\vec{x}_1, \ldots, \vec{x}_n$ are linearly indep.

$\det(X_1, X_2, X_3) = \begin{vmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{vmatrix} \begin{matrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{matrix} = \boxed{0}$

$\boxed{\text{Linearly dependent}}$

Since they are linearly dependent, then there exist constants a and b such that

$\vec{x}_1 = a\vec{x}_2 + b\vec{x}_3$

$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = a\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$

$\begin{cases} 2 = -b \\ 1 = a + 2b \end{cases}$ solve for a and b

$b = -2$

$a = 1 - 2b$

$= 5$

$\boxed{\vec{x}_1 = 5\vec{x}_2 - 2\vec{x}_3}$

(b) $X_1 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}.$

if $\vec{x}_1$ and $\vec{x}_2$ are linearly dependent, then there exists a constant c such that $\vec{x}_1 = c\,\vec{x}_2$

$$\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = c \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}$$ not possible.

$\boxed{\text{linearly independent}}$

$\vec{x}_1(t)$ and $\vec{x}_2(t)$ are linearly independent if

$c_1\vec{x}_1(t)+c_2\vec{x}_2(t)=\vec{0}$ if and only if $c_1=c_2=0$.

for all t.

7. Are the vector functions linearly independent? If they are linearly dependent, find a linear relation among them.

(a) $X_1(t) = \begin{pmatrix} e^{-3t} \\ -4e^{-3t} \end{pmatrix} \qquad X_2(t) = \begin{pmatrix} e^{-3t} \\ e^{-3t} \end{pmatrix}.$

$$c_1\vec{x}_1(t)+c_2\vec{x}_2(t)=c_1\begin{pmatrix} e^{-3t} \\ -4e^{-3t} \end{pmatrix}+c_2\begin{pmatrix} e^{-3t} \\ e^{-3t} \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} [c_1e^{-3t}+c_2e^{-3t}=0]\,e^{3t} \\ [-4c_1e^{-3t}+c_2e^{-3t}=0]\,e^{3t} \end{cases}$$

$$\begin{cases} c_1+c_2=0 \\ -4c_1+c_2=0 \end{cases} \qquad c_2=-c_1$$

$$-5c_1=0 \Rightarrow \boxed{c_1=c_2=0}$$

$\boxed{\text{linearly independent}}$

(b) $X_1(t) = \begin{pmatrix} 2e^t \\ -e^t \end{pmatrix}$, $X_2(t) = \begin{pmatrix} e^t \\ 2e^t \end{pmatrix}$ $X_3(t) = \begin{pmatrix} 3te^t \\ te^t \end{pmatrix}$.

linearly independent.

8. Find the eigenvalues and eigenvectors of the given matrix

(a) $A = \begin{pmatrix} -2 & 1 \\ -1 & -2 \end{pmatrix}$

Find eigenvalues: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} -2-\lambda & 1 \\ -1 & -2-\lambda \end{vmatrix} = (-2-\lambda)(-2-\lambda) + 1$$
$$= 4 + 4\lambda + \lambda^2 + 1$$
$$= \lambda^2 + 4\lambda + 5 = 0$$

solve for λ.

$$\lambda_1 = \frac{-4 + \sqrt{16-20}}{2} = \frac{-4+2i}{2} = \boxed{-2+i}$$

$$\lambda_2 = \overline{\lambda_1} = \boxed{-2-i}$$

Eigenvectors. $\lambda_1 = -2+i$

$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is an eigenvector that corresponds to λ_1

$\vec{v}$ is a solution to the system

$$(A - \lambda_1 I)\vec{v} = \vec{0}$$

$$\begin{pmatrix} -2-(-2+i) & 1 \\ -1 & -2-(-2+i) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -i v_1 + v_2 = 0 \\ -v_1 - i v_2 = 0 \end{cases}$$

(1st eqn) = (2nd eqn.)(i)

$$v_2 = i v_1$$

$$\vec{v}_1 = \begin{pmatrix} v_1 \\ i v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ i \end{pmatrix} \overset{v_1 = 1}{=} \boxed{\begin{pmatrix} 1 \\ i \end{pmatrix} \text{ corresponds to } \lambda_1 = -2+i}$$

vector $\vec{v}_2 = \overline{(\vec{v}_1)} = \boxed{\begin{pmatrix} 1 \\ -i \end{pmatrix} \text{ corresponds to } \lambda_2 = -2-i}$

(b) $A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$.

eigenvalues! $\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix}$

$$= (3-\lambda)(-\lambda)(3-\lambda) + 16 + 16 + 16\lambda - 4(3-\lambda) - 4(3-\lambda)$$

$$= -\lambda^3 + 6\lambda^2 + 15\lambda + 8 = 0$$

$$\lambda^3 - 6\lambda^2 - 15\lambda - 8 = 0$$

$\lambda = 1$: $\quad 1-6-15-8 \neq 0$

eigenvalue $\lambda = -1$: $\quad -1-6+15-8 = 0$

$$\lambda^3 - 6\lambda^2 - 15\lambda - 8 = (\lambda+1)(\lambda^2 - 7\lambda - 8)$$

$$= (\lambda+1)(\lambda-8)(\lambda+1)$$

$$= (\lambda+1)^2(\lambda-8) = 0$$

$\lambda_1 = 8$

$\lambda_2 = -1$ - repeated root.

$$\begin{array}{r|l} & \lambda^2 - 7\lambda - 8 \\ \hline \lambda+1 & \lambda^3 - 6\lambda^2 - 15\lambda - 8 \\ & \lambda^3 + \lambda^2 \\ \hline & -7\lambda^2 - 15\lambda \\ & -7\lambda^2 - 7\lambda \\ \hline & -8\lambda - 8 \end{array}$$

Eigenvectors. $\lambda_1 = 8$ $\quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$\vec{v}$ is a solution of $(A - \lambda_1 I)\vec{v} = \vec{0}$

$$\begin{pmatrix} 3-8 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & 3-8 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

solve for v_1, v_2, v_3

$$\begin{cases} -5v_1 + 2v_2 + 4v_3 = 0 \\ 2v_1 - 8v_2 + 2v_3 = 0 \\ 4v_1 + 2v_2 - 5v_3 = 0 \end{cases} \Rightarrow \begin{cases} -5v_1 + 2v_2 + 4v_3 = 0 \\ v_1 - 4v_2 + v_3 = 0 \\ 4v_1 + 2v_2 - 5v_3 = 0 \end{cases}$$

1st + 3rd $\begin{cases} v_1 - 4v_2 + v_3 = 0 \\ -v_1 + 4v_2 - v_3 = 0 \\ 4v_1 + 2v_2 - 5v_3 = 0 \end{cases}$

$$\begin{cases} v_1 - 4v_2 + v_3 = 0 \\ 4v_1 + 2v_2 - 5v_3 = 0 \end{cases}$$

$$v_1 = 4v_2 - v_3$$

$$4(4v_2 - v_3) + 2v_2 - 5v_3 = 0$$

$$16v_2 - 4v_3 + 2v_2 - 5v_3 = 0$$

$$18v_2 - 9v_3 = 0$$

$$\boxed{v_3 = 2v_2}$$

$$v_1 = 4v_2 - 2v_2 = 2v_2$$

$$\vec{v} = \begin{pmatrix} 2v_2 \\ v_2 \\ 2v_2 \end{pmatrix} = v_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \overset{v_2=1}{=} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$ corresponds to $\lambda_1 = 8$

$\lambda_2 = -1$ eigenvector $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ is a solution of $(A - \lambda_2 I)\vec{w} = \vec{0}$

$$\begin{pmatrix} 3-(-1) & 2 & 4 \\ 2 & -(-1) & 2 \\ 4 & 2 & 3-(-1) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 4w_1 + 2w_2 + 4w_3 = 0 \\ 2w_1 + w_2 + 2w_3 = 0 \\ 4w_1 + 2w_2 + 4w_3 = 0 \end{cases} \Rightarrow \begin{array}{l} 2w_1 + w_2 + 2w_3 = 0 \\ w_2 = -2w_1 - 2w_3 \end{array}$$

$$\vec{w} = \begin{pmatrix} w_1 \\ -2w_1 - 2w_3 \\ w_3 \end{pmatrix} = w_1 \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + w_3 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

$w_1 = 1, w_3 = 0$ $\quad \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = \vec{w_1}$

$w_1 = 0, w_3 = 1$ $\quad \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} = \vec{w_2}$

correspond to $\lambda = -1$

(c) $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix}.$