1. Find the general solution of the system. Classify the critical point (0,0) as to type, determine whether it is stable or unstable, sketch the phase portrait.

(a) $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\mathbf{x}$, $A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$, $tr(A) = 3-2 = 1$, $\det A = -6+4 = -2$

Characteristic equation $\det(A - \lambda I) = 0$

$\lambda^2 - (tr\, A)\lambda + (\det A) = 0.$

$\lambda^2 - \lambda - 2 = 0$

$(\lambda - 2)(\lambda + 1) = 0$

$\boxed{\lambda_1 = 2,\ \lambda_2 = -1}$ eigenvalues.

Eigenvectors

$\lambda_1 = 2$, $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ it is a solution of the system

$(A - 2I)\vec{v} = \vec{0}$ or $\begin{pmatrix} 3-2 & -2 \\ 2 & -2-2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ or $\begin{cases} v_1 - 2v_2 = 0 \\ 2v_1 - 4v_2 = 0 \end{cases} \Rightarrow \boxed{v_1 = 2v_2}$

update $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2v_2 \\ v_2 \end{pmatrix} \overset{v_2=1}{=} \boxed{\begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ corresponds } \lambda = 2}$

$\lambda_2 = -1$ $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$, it is a solution of $(A - (-1)I)\vec{w} = \vec{0}$ or $(A + I)\vec{w} = \vec{0}$

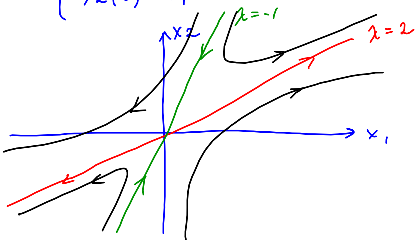
$\begin{pmatrix} 3+1 & -2 \\ 2 & -2+1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{cases} 4w_1 - 2w_2 = 0 \\ 2w_1 - w_2 = 0 \end{cases} \Rightarrow w_2 = 2w_1$

update $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ 2w_1 \end{pmatrix} \overset{w_1=1}{=} \boxed{\begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ corresponds to } \lambda = -1}$

General solution: $\boxed{\vec{x}(t) = C_1\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{2t} + C_2\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{-t}}$ $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 2C_1e^{2t} + C_2e^{-t} \\ C_1e^{2t} + 2C_2e^{-t} \end{pmatrix}$

Phase portrait $\begin{cases} x_1(t) = 2C_1e^{2t} + C_2e^{-t} \\ x_2(t) = C_1e^{2t} + 2C_2e^{-t} \end{cases}$



(0,0) is a saddle point

it is unstable.

(b) $\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \mathbf{x}$

$A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}$, $tr(A) = 1 - 4 = -3$

$det(A) = -4 + 6 = 2$

<u>Characteristic equation</u>

$\lambda^2 - (tr\, A)\lambda + det\, A = 0$

$\lambda^2 + 3\lambda + 2 = 0$

$(\lambda+2)(\lambda+1) = 0$

$\boxed{\lambda_1 = -2 \quad \lambda_2 = -1}$ eigenvalues.

<u>Eigenvectors</u>

<u>$\lambda_1 = -2$</u>, $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, it is a solution $(A - (-2)I)\vec{v} = \vec{0}$ or $(A + 2I)\vec{v} = \vec{0}$

$$\begin{pmatrix} 1+2 & -2 \\ 3 & -4+2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -2 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{matrix} 3v_1 - 2v_2 = 0 \\ 3v_1 = 2v_2 \end{matrix} \quad \text{or} \quad v_1 = \frac{2v_2}{3}$$

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \frac{2v_2}{3} \\ v_2 \end{pmatrix} \underset{v_2 = 3}{=} \boxed{\begin{pmatrix} 2 \\ 3 \end{pmatrix} \text{ corresponds to } \lambda_1 = -2}$$

<u>$\lambda_2 = -1$</u>, $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$, it is a solution of $(A - (-1)I)\vec{w} = \vec{0}$ or $(A + I)\vec{w} = \vec{0}$

$$\begin{pmatrix} 1+1 & -2 \\ 3 & -4+1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 2 & -2 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{cases} 2w_1 - 2w_2 = 0 \\ 3w_1 - 3w_2 = 0 \end{cases}$$

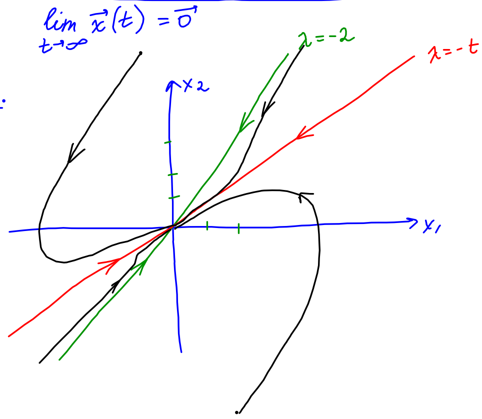
or $w_1 = w_2$

$$\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_1 \end{pmatrix} \underset{w_1 = 1}{=} \boxed{\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ corresponds to } \lambda_2 = -1}$$

<u>General solution</u> $\boxed{\vec{x}(t) = c_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}}$

$\lim_{t \to \infty} \vec{x}(t) = \vec{0}$

<u>Phase portrait.</u>



$\vec{x}(t) = e^{-t}\left[c_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]$

(0,0) is a <u>nodal sink</u>

it is <u>asymptotically stable</u>

(c) $\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}\mathbf{x}$, $A = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}$, $tr(A) = 5+1=6$
$\det(A) = 5+3=8$

Characteristic equation $\lambda^2 - 6\lambda + 8 = 0$

$(\lambda-4)(\lambda-2) = 0$

$\boxed{\lambda_1 = 4;\ \lambda_2 = 2}$ eigenvalues

Eigenvectors $\lambda_1 = 4$, $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is a solution of $(A-4I)\vec{v} = \vec{0}$

$$\begin{pmatrix} 5-4 & -1 \\ 3 & 1-4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ or } \begin{matrix} v_1 - v_2 = 0 \\ v_1 = v_2 \end{matrix}$$

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_1 \end{pmatrix} \overset{v_1=1}{=} \boxed{\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ corresponds to } \lambda_1 = 4}$$

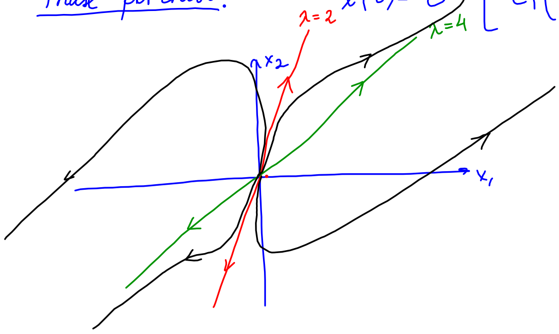
$\lambda_2 = 2$ $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ is a solution of $(A-2I)\vec{w} = \vec{0}$

$$\begin{pmatrix} 5-2 & -1 \\ 3 & 1-2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ or } \begin{matrix} 3w_1 - w_2 = 0 \\ w_2 = 3w_1 \end{matrix}$$

$$\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ 3w_1 \end{pmatrix} \overset{w_1=1}{=} \boxed{\begin{pmatrix} 1 \\ 3 \end{pmatrix} \text{ corresponds to } \lambda_2 = 2}$$

General solution: $\boxed{\vec{x}(t) = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{4t} + C_2\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{2t}}$

Phase portrait: $\vec{x}(t) = e^{4t}\left[C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{-2t}\right]$



(0,0) is a nodal source

it is asymptotically unstable

(d) $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ \boxed{1} & -2 \end{pmatrix}\mathbf{x}$, $A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$, $\operatorname{tr} A = 0$, $\det A = -4+5 = 1$

CCW

Characteristic equation $\lambda^2 + 1 = 0$

$\lambda^2 = -1$ or $\lambda = \pm i$

Eigenvectors $\lambda_1 = i$, $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is a solution of $(A - iI)\vec{v} = \vec{0}$

$$\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{cases} (2-i)v_1 - 5v_2 = 0 \\ v_1 - (2+i)v_2 = 0 \end{cases} \Rightarrow \quad v_1 = (2+i)v_2$$

$$[v_1 - (2+i)v_2](2-i) = 0 \Rightarrow (2-i)v_1 - (2^2 - i^2)v_2 = 0$$
$$(2-i)v_1 - 5v_2 = 0.$$

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} (2+i)v_2 \\ v_2 \end{pmatrix} \overset{v_2=1}{=} \boxed{\begin{pmatrix} 2+i \\ 1 \end{pmatrix} \text{ corresponds to } \lambda_1 = i}$$

$$\vec{v}e^{\lambda t} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{it} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}(\cos t + i \sin t) = \begin{pmatrix} (2+i)(\cos t + i\sin t) \\ \cos t + i \sin t \end{pmatrix}$$

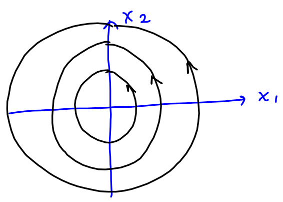
$$= \begin{pmatrix} 2\cos t + 2i\sin t + i\cos t + \overset{-1}{i^2}\sin t \\ \cos t + i\sin t \end{pmatrix} = \begin{pmatrix} 2\cos t - \sin t + i(2\sin t + \cos t) \\ \cos t + i \sin t \end{pmatrix}$$

$$= \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + \begin{pmatrix} i(2\sin t + \cos t) \\ i \sin t \end{pmatrix} = \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + i\begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}$$

General solution $\vec{x}(t) = c_1 \operatorname{Re}(\vec{v}e^{\lambda t}) + c_2 \operatorname{Im}(\vec{v}e^{\lambda t})$

$$\boxed{\vec{x}(t) = c_1\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2\begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}}$$

Phase portrait



(0,0) is a center

it is stable

(e) $\mathbf{x}' = \begin{pmatrix} 1 & -5 \\ \boxed{1} & -3 \end{pmatrix} \mathbf{x}$, $A = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix}$, $tr(A) = 1-3 = -2$, $det(A) = -3+5 = 2$

CCW

Characteristic equation

$$\lambda^2 + 2\lambda + 2 = 0$$

$$\lambda_1 = \frac{-2+\sqrt{4-8}}{2} = \frac{-2+\sqrt{-4}}{2} = \frac{-2+2i}{2} = -1+i$$

$$\lambda_2 = \bar{\lambda}_1 = -1-i$$

Eigenvectors $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is a solution of $(A - (-1+i)I)\vec{v} = \vec{0}$

$$\begin{pmatrix} 1-(-1+i) & -5 \\ 1 & -3-(-1+i) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ or } \begin{cases} (2-i)v_1 - 5v_2 = 0 \\ v_1 - (2+i)v_2 = 0 \end{cases} \Longrightarrow v_1 = (2+i)v_2$$

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} (2+i)v_2 \\ v_2 \end{pmatrix} \underset{v_2=1}{=} \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$$

$$\vec{v}e^{\lambda t} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{(-1+i)t} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{-t}(\cos t + i \sin t)$$

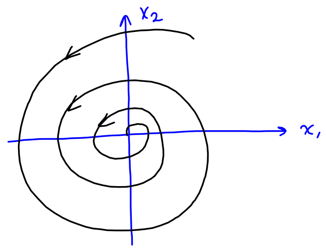
$$= e^{-t}\left[\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + i \begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix} \right]$$

$$e^{a+ib} = e^a(\cos b + i \sin b)$$

General solution

$$\boxed{\vec{x}(t) = e^{-t}\left[c_1 \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix} \right]}$$

Phase portrait



(0,0) is a spiral sink

it is asymptotically stable

(f) $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ \boxed{4} & -1 \end{pmatrix}\mathbf{x}$

CCW

$A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$, $tr(A) = 3-1 = 2$

$det(A) = -3+8 = 5$

Characteristic equation: $\lambda^2 - 2\lambda + 5 = 0$

$$\lambda_1 = \frac{2+\sqrt{4-20}}{2} = \frac{2+\sqrt{-16}}{2} = \frac{2+4i}{2} = 1+2i$$

$$\lambda_2 = \overline{\lambda_1} = 1-2i$$

Eigenvector $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$$(A-(1+2i)I)\vec{v} = \vec{0}$$

$$\begin{pmatrix} 3-(1+2i) & -2 \\ 4 & -1-(1+2i) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2-2i & -2 \\ 4 & -2-2i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{cases} (2-2i)v_1 - 2v_2 = 0 \Rightarrow v_2 = (1-i)v_1 \\ 4v_1 - (2+2i)v_2 = 0 \end{cases}$$

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ (1-i)v_1 \end{pmatrix} \overset{v_1=1}{=} \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$$

$$\vec{v}e^{\lambda t} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix} e^{(1+2i)t} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix} e^t(\cos 2t + i\sin 2t)$$

$$= e^t\begin{pmatrix} \cos 2t + i\sin 2t \\ (1-i)(\cos 2t + i\sin 2t) \end{pmatrix} = e^t\begin{pmatrix} \cos 2t + i\sin 2t \\ \cos 2t + i\sin 2t - i\cos 2t - \overset{-1}{\cancel{i^2}}\sin 2t \end{pmatrix}$$

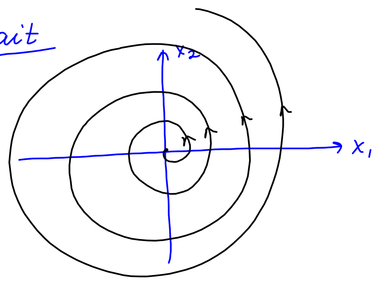
$$= e^t\begin{pmatrix} \cos 2t + i\sin 2t \\ \cos 2t + \sin 2t + i(\sin 2t - \cos 2t) \end{pmatrix}$$

$$= e^t\left[\begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + i\begin{pmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{pmatrix}\right]$$

General solution

$$\boxed{\vec{x}(t) = e^t\left[c_1\begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + c_2\begin{pmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{pmatrix}\right]}$$

Phase portrait



$(0,0)$ is a spiral source

it is unstable

2. Solve the initial value problem.

(a) $\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

General solution (see #1c)

$$\vec{x}(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + C_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}$$

$$\vec{x}(0) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\begin{cases} C_1 + C_2 = 2 \implies C_1 = 2 - C_2 = 2 - \left(-\frac{3}{2}\right) = \frac{7}{2} \\ C_1 + 3C_2 = -1 \end{cases}$$

$$C_2 - 3C_2 = 2 - (-1) \implies -2C_2 = 3 \implies C_2 = -\frac{3}{2}$$

$$\vec{x}(t) = \frac{7}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} - \frac{3}{2}\begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}$$

or

$$\boxed{\begin{cases} x_1(t) = \frac{7}{2} e^{4t} - \frac{3}{2} e^{2t} \\ x_2(t) = \frac{7}{2} e^{4t} - \frac{9}{2} e^{2t} \end{cases}}$$

(b) $\mathbf{x}' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix}\mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

General solution (see #1e)

$$\boxed{\vec{x}(t) = e^{-t}\left[c_1\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2\begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}\right]}$$

$$\vec{x}(0) = c_1\begin{pmatrix} 2\cancelto{1}{\cos 0} - \cancelto{0}{\sin 0} \\ \cancelto{1}{\cos 0} \end{pmatrix} + c_2\begin{pmatrix} 2\cancelto{0}{\sin 0} + \cancelto{1}{\cos 0} \\ \cancelto{0}{\sin 0} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{cases} 2c_1 + c_2 = 1 \\ c_1 = 1 \end{cases} \Rightarrow c_2 = 1 - 2c_1 = 1 - 2 = -1$$

$$\vec{x}(t) = e^{-t}\left[\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} - \begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}\right]$$

$$= e^{-t}\begin{pmatrix} 2\cos t - \sin t - 2\sin t - \cos t \\ \cos t - \sin t \end{pmatrix}$$

$$= e^{-t}\begin{pmatrix} \cos t - 3\sin t \\ \cos t - \sin t \end{pmatrix} \quad \text{or} \quad \boxed{\begin{cases} x_1(t) = e^{-t}(\cos t - 3\sin t) \\ x_2(t) = e^{-t}(\cos t - \sin t) \end{cases}}$$

3. Find the general solution of the system.

(a) $\mathbf{x}' = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \mathbf{x}.$

Characteristic equation : $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} \begin{matrix} 1-\lambda & -1 \\ 3 & 2-\lambda \\ 2 & 1 \end{matrix}$$

$= (1-\lambda)(2-\lambda)(-1-\lambda) + 2 + 12 - 3(1+\lambda) + (1-\lambda) - 8(2-\lambda)$

$= -(1-\lambda)(2-\lambda)(1+\lambda) + 14 - 3 - 3\lambda + 1 - 16 + 8\lambda - \lambda$

$= -(1-\lambda^2)(2-\lambda) - 4 + 4\lambda$

$= -2 + \lambda + 2\lambda^2 - \lambda^3 - 4 + 4\lambda$

$= -\lambda^3 + 2\lambda^2 + 5\lambda - 6 = 0$

$\boxed{\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0}$

$(\lambda - 1)(\lambda^2 - \lambda - 6) = 0$

$(\lambda - 1)(\lambda - 3)(\lambda + 2) = 0$

$\boxed{\lambda_1 = 1,\ \lambda_2 = 3,\ \lambda_3 = -2}$

eigen values.

$$\begin{array}{r|l} & \lambda^2 - \lambda - 6 \\ \hline \lambda - 1 & \lambda^3 - 2\lambda^2 - 5\lambda + 6 \\ & \lambda^3 - \lambda^2 \\ \hline & -\lambda^2 - 5\lambda \\ & -\lambda^2 + \lambda \\ \hline & -6\lambda + 6 \end{array}$$

Eigenvectors $\lambda_1 = 1$, $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ is a solution of $(A - I)\vec{v} = \vec{0}$

$$\begin{pmatrix} 1-1 & -1 & 4 \\ 3 & 2-1 & -1 \\ 2 & 1 & -1-1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -v_2 + 4v_3 = 0 \implies v_2 = 4v_3 \\ 3v_1 + v_2 - v_3 = 0 \\ 2v_1 + v_2 - 2v_3 = 0 \end{cases}$$

$3v_1 + 4v_3 - v_3 = 0$

$3v_1 + 3v_3 = 0 \implies v_1 = -v_3$

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -v_3 \\ 4v_3 \\ v_3 \end{pmatrix} \overset{v_3 = 1}{=} \boxed{\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} \text{ corresponds to } \lambda_1 = 1}$$

$$\boxed{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \text{ corresponds to } \lambda_2 = 3}$$

$$\boxed{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \text{ corresponds to } \lambda = -2}$$

General solution : $\boxed{\vec{x}(t) = C_1 \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} e^{t} + C_2 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} e^{3t} + C_3 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} e^{-2t}}$

(b) $\mathbf{x}' = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix} \mathbf{x}.$

Eigenvalues: $\lambda_1 = 1, \quad \lambda_2 = 1+2i, \quad \lambda_3 = 1-2i$

Eigenvectors: $\vec{v} = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}, \ \lambda_1 = 1$

$\vec{w} = \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix}, \ \lambda = 1+2i$

$$\vec{w} e^{\lambda_2 t} = \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix} e^{(1+2i)t} = \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix} e^t (\cos 2t + i \sin 2t)$$

$$= e^t \begin{pmatrix} 0 \\ \cos 2t + i \sin 2t \\ -i\cos 2t - \cancel{i^2}^{-1} \sin 2t \end{pmatrix} = e^t \begin{pmatrix} 0 \\ \cos 2t + i \sin 2t \\ \sin 2t - i \cos 2t \end{pmatrix}$$

$$= e^t \left[\begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t \end{pmatrix} + i \begin{pmatrix} 0 \\ \sin 2t \\ \cos 2t \end{pmatrix} \right]$$

General solution

$$\boxed{\vec{x}(t) = C_1 e^t \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} + C_2 e^t \begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t \end{pmatrix} + C_3 e^t \begin{pmatrix} 0 \\ \sin 2t \\ \cos 2t \end{pmatrix}}$$