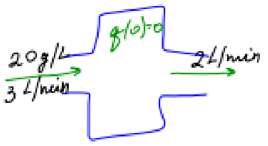
1. A large tank initially contains 10 L of fresh water. A brine containing 20 g/L of salt flows into the tank at a rate of 3 L/min. The solution inside the tank is kept well stirred and flows out of the tank at the rate 2 L/min. Determine the concentration of salt in the tank as a function of time.



$q(t)$ is the mass @ tank @ time t

$$\frac{dq}{dt} = \text{rate in} - \text{rate out}$$

$$= 20(3) - (2)\,\frac{q(t)}{10+(3-2)t}$$

$$\frac{dq}{dt} = 60 - \frac{2q}{10+t}, \quad q(0)=0$$

$$\frac{dq}{dt} + \frac{2}{10+t}\,q = 60 \qquad \text{linear, } \quad P(t)=\frac{2}{10+t}, \quad Q(t)=60$$

Integrating factor μ : $\dfrac{d\mu}{dt} = \dfrac{2}{10+t}\mu$ separable

$$\int \frac{d\mu}{\mu} = \int \frac{2}{10+t}\,dt$$

$$\ln|\mu| = 2\ln|10+t| \Rightarrow \mu(t) = (10+t)^2$$

$$\mu(t)\,q(t) = \int 60\,\mu(t)\,dt$$

$$(10+t)^2 q(t) = 60\int (10+t)^2\,dt$$

$$\frac{(10+t)^2 q(t)}{(10+t)^2} = \frac{20(10+t)^3 + C}{(10+t)^2}$$

$$q(t) = 20(10+t) + \frac{C}{(10+t)^2}$$

$$q(0) = 200 + \frac{C}{100} = 0 \Rightarrow C = -20000$$

$$q(t) = 20(10+t) - \frac{20000}{(10+t)^2} \quad \text{- mass}$$

$$\boxed{\frac{q(t)}{10+t} = 20 - \frac{20000}{(10+t)^3}} \quad \text{concentration}$$

2. An object with temperature 150° is placed in a freezer whose temperature is 30°. Assume that the temperature of the freezer remains essentially constant.

(a) If the object is cooled to 120° after 8 min, what will its temperature be after 18 min?

(b) When will its temperature be 60°?

$T(t)$ is the temperature of the object @ time t.

$$\boxed{\frac{dT}{dt} = k(M-T)}, \quad M \text{ is the outside temperature. } k \text{ is an unknown constant}$$

$M = 30$

$T(0) = 150$, $T(8) = 120$, $T(18) = ?$

time t such that $T(t) = 60$

$$\frac{dt\,\frac{dT}{dt}}{T-30} = \frac{-k(30+T)\,dt}{T-30} \Rightarrow \int \frac{dT}{T-30} = \int -k\,dt$$

$$\ln|T-30| = -kt + C$$

$$T - 30 = C_1 e^{-kt}, \quad C_1 = e^C$$

$$T = 30 + C_1 e^{-kt}$$

$$T(0) = 30 + C_1 = 150 \Rightarrow T(t) = 30 + 120 e^{-kt}$$

$$C_1 = 120$$

$$T(8) = 30 + 120 e^{-8k} = 120$$

$$12\cancel{0} e^{-8k} = 9\cancel{0}$$

$$e^{-8k} = \frac{9}{12} = \frac{3}{4} \Rightarrow -8k = \ln\frac{3}{4} \Rightarrow k = -\frac{1}{8}\ln\frac{3}{4}$$

$$\boxed{T(t) = 30 + 120 e^{\frac{t}{8}\ln\frac{3}{4}}}$$

$$T(18) = 30 + 120 e^{\frac{18}{8}\ln\frac{3}{4}}$$

Find t such that $T(t) = 30 + 120 e^{\frac{t}{8}\ln\frac{3}{4}} = 60$

$$120 e^{\frac{t}{8}\ln\frac{3}{4}} = 30$$

$$e^{\frac{t}{8}\ln\frac{3}{4}} = \frac{1}{4}$$

$$\frac{t}{8}\ln\frac{3}{4} = \ln\frac{1}{4}$$

$$\boxed{t = 8\,\frac{\ln\frac{1}{4}}{\ln\frac{3}{4}}}$$

3. Determine (without solving the problem) an interval in which the solution to the initial value problem

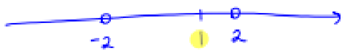
$$(4 - t^2)y' + 2ty = 3t^2, \qquad y(1) = -3$$

is certain to exist.

$$\frac{(4 - t^2)y' + 2ty = 3t^2,}{(4-t^2)} \qquad y(1) = -3$$

is certain to exist.

$$y' + \frac{2t}{4-t^2}\,y = \frac{3t^2}{4-t^2}, \qquad 4-t^2 \neq 0, \quad t \neq \pm 2$$



$$\boxed{(-2,2)}$$

$\sqrt{t} \Rightarrow t \geq 0$	$\cot t = \frac{\cos t}{\sin t} \Rightarrow t \neq 0, \pm\pi, \pm 2\pi, \pm 3\pi \ldots$
$\ln t \rightarrow t > 0$	$\tan t = \frac{\sin t}{\cos t} \Rightarrow t \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \ldots$
$\frac{1}{\ln t} \rightarrow t > 0, \quad t \neq 1$	

4. Solve the initial value problem

$$y' = \frac{t^2}{1+t^3}, \qquad y(0) = y_0$$

and determine how the interval in which the solution exists depends on the initial value y_0.

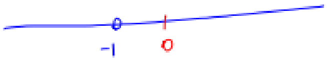
$y' + p(t)y = g(t)$

4. Solve the initial value problem

$$y' = \frac{t^2}{1+t^3}, \qquad y(0) = y_0$$

and determine how the interval in which the solution exists depends on the initial value y_0.

$g(t) = f(y,t) = \dfrac{t^2}{1+t^3}$ continuous for all $t \neq -1$



$\boxed{(-1, \infty)}$ solution on

$$\frac{dy}{dt} = \frac{t^2}{1+t^3} \Rightarrow dy = \frac{t^2}{1+t^3}\,dt$$

$$y = \frac{1}{3}\ln|1+t^3| + C$$

$$y(0) = \frac{1}{3}\ln 1 + C = C = y_0$$

solution of the initial value problem:

$\boxed{y = \frac{1}{3}\ln|1+t^3| + y_0}$ exists for all y_0

5. Solve the following initial value problem

$$\sqrt{y}dt + (1+t)dy = 0 \quad y(0) = 1.$$

5. Solve the following initial value problem

$$\sqrt{y}dt + (1+t)dy = 0 \quad y(0) = 1.$$

not linear
not exact
separable.

$$\sqrt{y}\, dt = -(1+t)dy$$

$$\int \frac{dy}{\sqrt{y}} = -\int \frac{dt}{1+t}$$

$$\frac{y^{1/2}}{1/2} = -\ln|1+t| + C$$

$$2\sqrt{y} = C - \ln|1+t|$$

$$\sqrt{y} = \frac{C - \ln|1+t|}{2}$$ plug in $y=1$ and $t=0$

$$1 = \frac{C - \ln 1}{2} \quad \text{or} \quad C = 2$$

$$\boxed{\sqrt{y} = \frac{2 - \ln|1+t|}{2}}$$ solution of the initial value problem.

6. Find the general solution to the equation

$$(t^2 - 1)y' + 2ty + 3 = 0$$

6. Find the general solution of the equation

$$(t^2 - 1)y' + 2ty + 3 = 0 \qquad \text{linear}$$

$$y' + \frac{2t}{t^2-1} y + \frac{3}{t^2-1} = 0$$

$$p(t) = \frac{2t}{t^2-1}, \quad g(t) = -\frac{3}{t^2-1}$$

$$\frac{d\mu}{dt} = \frac{2t}{t^2-1}\mu \Rightarrow \mu = t^2-1$$

$$(t^2-1)y = \int -\frac{3}{t^2-1}(t^2-1)\, dt = -3t + C$$

$$\boxed{y = -\frac{3t}{t^2-1} + \frac{C}{t^2-1}}$$

7. Given the differential equation

$$\frac{dy}{dt} = 7y - y^2 - 10$$

(a) Find the equilibrium solutions

(b) Sketch the phase line and determine whether the equilibrium solutions are stable, unstable, or semistable

(c) Graph some solutions

(d) If $y(t)$ is the solution of the equation satisfying the initial condition $y(0) = y_0$, where $-\infty < y_0 < \infty$, find the limit of $y(t)$ when $t \to \infty$

(e) Solve the equation

7a) $7y - y^2 - 10 = 0 \Rightarrow y^2 - 7y + 10 = 0.$

$(y-5)(y-2) = 0$

$y_1 = +5, \; y_2 = 2$

7b) $f(y) = 7y - y^2 - 10$

$f(0) = -10 < 0$

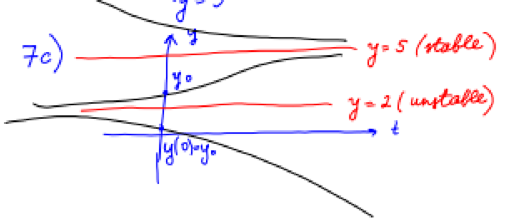
$f(3) = 7(3) - 9 - 10 > 0$

$f(6) = 42 - 36 - 10 < 0$

$y = 2$ unstable

$y = 5$ stable

7c)



7d) $y(0) = y_0.$

$y_0 < 2$	$2 < y_0 < 5$	$y_0 > 5$
$\lim_{t\to\infty} y(t) = -\infty$	$\lim_{t\to\infty} y(t) = 5$	$\lim_{t\to\infty} y(t) = 5$
$\lim_{t\to-\infty} y(t) = 2$	$\lim_{t\to-\infty} y(t) = 2$	$\lim_{t\to-\infty} y(t) = \infty$

$\frac{dy}{dt} = 7y - y^2 - 10$

$\frac{dy}{dt} = -(y-5)(y-2)$

$\frac{dy}{(y-5)(y-2)} = -dt$

$\frac{1}{(y-5)(y-2)} = \frac{1}{3}\left[\frac{1}{y-5} - \frac{1}{y-2}\right]$

$\frac{1}{3}\left[\ln|y-5| - \ln|y-2|\right] = -t + C$

$\boxed{\ln\left|\frac{y-5}{y-2}\right| = -3t + C_1}$

$(uvw)' = u'vw + uv'w + uvw'$

8. Solve the initial value problem

$$\underbrace{(ye^{xy}\cos(2x) - 2e^{xy}\sin(2x) + 2x)}_{M}dx + \underbrace{(xe^{xy}\cos(2x) - 3)}_{N}dy = 0, \qquad y(0) = -1$$

$\frac{\partial M}{\partial y} = e^{xy}\cos(2x) + y\,e^{xy}(x)\cos(2x) - 2e^{xy}(x)\sin(2x)$ match

$\frac{\partial N}{\partial x} = e^{xy}\cos(2x) + x\,e^{xy}\,y\,\cos(2x) + xe^{xy}(-2)\sin(2x)$

exact

$F(x,y)$: $\begin{cases} \frac{\partial F}{\partial x} = ye^{xy}\cos(2x) - 2e^{xy}\sin(2x) + 2x \\ \int \frac{\partial F}{\partial y}dy \int [xe^{xy}\cos(2x) - 3]\,dy \end{cases}$

$\int e^{xy}\,dy = \frac{1}{x}e^{xy} + C$

$F(x,y) = \frac{x}{x}e^{xy}\cos 2x - 3y + g(x)$

$F(x,y) = e^{xy}\cos 2x - 3y + g(x)$

$\frac{\partial F}{\partial x} = e^{xy}\cancel{(y)\cos 2x} + \cancel{e^{xy}(-2)\sin(2x)} + g'(x) = \cancel{ye^{xy}\cos(2x)} - \cancel{2e^{xy}\sin(2x)} + 2x$

$g'(x) = 2x$ or $g(x) = x^2 + C$

$F(x,y) = e^{xy}\cos 2x - 3y + x^2 + C$

General solution: $\boxed{e^{xy}\cos 2x - 3y + x^2 + C = 0}$

9. Find an integrating factor for the equation

$$\underbrace{(3xy + y^2)}_{M} + \underbrace{(x^2 + xy)}_{N}y' = 0$$

and then solve the equation.

$$\frac{M_y - N_x}{N} = \frac{(3x+2y)-(2x+y)}{x^2+xy} = \frac{x+y}{x(x+y)} \to \frac{1}{x} \quad \text{depends on } x \text{ only}$$

Integrating factor $\mu(x)$: $\quad \frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu$

$$\frac{d\mu}{dx} = \frac{\mu}{x}$$

$$\frac{d\mu}{\mu} = \frac{dx}{x} \Rightarrow \ln|\mu| = \ln|x| \Rightarrow \mu(x) = x$$

$$x(3xy+y^2) + x(x^2+xy)y' = 0 \quad , \quad y' = \frac{dy}{dx}$$

$$\underbrace{(3x^2y + xy^2)}_{M(x,y)}dx + \underbrace{(x^3+x^2y)}_{N(x,y)}dy = 0$$

$$\frac{\partial M}{\partial y} = 3x^2 + 2xy \qquad \frac{\partial N}{\partial x} = 3x^2 + 2xy \qquad \text{exact.}$$

$$F(x,y): \begin{cases} \frac{\partial F}{\partial x} = 3x^2y + xy^2 \\ \int \frac{\partial F}{\partial y} dy \int [x^3 + x^2y]\, dy \end{cases} \Rightarrow F(x,y) = x^3y + \frac{x^2y^2}{2} + g(x)$$

$$\frac{\partial F}{\partial x} = \cancel{3x^2y} + \cancel{\frac{2xy^2}{2}} + g'(x) = \cancel{3x^2y} + \cancel{xy^2}$$

$$g'(x) = 0$$

$$g(x) = C$$

$$F(x,y) = x^3y + \frac{x^2y^2}{2} + C$$

General solution: $\boxed{x^3y + \frac{x^2y^2}{2} + C = 0}$

10. Solve the equation/initial value problem

(a) $6y'' - 5y' + y = 0$, $y(0) = 4$, $y'(0) = 0$

(b) $4y'' - 12y' + 9y = 0$

(c) $y'' + 4y' + 5y = 0$, $y(0) = 0$, $y'(0) = 1$

a) $6y'' - 5y' + y = 0$, $y(0) = 4$, $y'(0) = 0$

auxiliary equation: $y'' \to r^2$, $y' \to r$, $y \to 1$

$6r^2 - 5r + 1 = 0$

$r_1 = \frac{5 + \sqrt{25-24}}{12} = \frac{1}{2}$

$r_2 = \frac{5-1}{12} = \frac{4}{12} = \frac{1}{3}$

General solution: $y(t) = C_1 e^{\frac{t}{2}} + C_2 e^{\frac{t}{3}}$
plug into the initial conditions

$y(t) = C_1 e^{\frac{t}{2}} + C_2 e^{\frac{t}{3}}$ | $y(0) = C_1 + C_2 = 0$

$y'(t) = \frac{C_1}{2} e^{\frac{t}{2}} + \frac{C_2}{3} e^{\frac{t}{3}}$ | $y'(0) = \frac{C_1}{2} + \frac{C_2}{3} = 1$

$\begin{cases} C_1 + C_2 = 0 \Rightarrow C_2 = -C_1 \\ 3C_1 + 2C_2 = 6 \end{cases}$

$3C_1 - 2C_1 = 6 \Rightarrow C_1 = 6,\ C_2 = -6$

$\boxed{y(t) = 6e^{\frac{t}{2}} - 6e^{\frac{t}{3}}}$

b) $4y'' - 12y' + 9y = 0$

$4r^2 - 12r + 9 = 0$

$(2r-3)^2 = 0$

$r = \frac{3}{2}$ repeated

$\boxed{y(t) = (C_1 + C_2 t)e^{\frac{3}{2}t}}$

c) $y'' + 4y' + 5y = 0$, $y(0) = 0$, $y'(0) = 1$

$r^2 + 4r + 5 = 0$

$r_1 = \frac{-4 + \sqrt{16-20}}{2} = \frac{-4 + \sqrt{-4}}{2} = \frac{-4+2i}{2}$

$r_1 = -2 + i$

$Re(r_1) = -2$, $Im(r_1) = 1$

General solution is

$\boxed{y(t) = e^{-2t}[C_1 \cos t + C_2 \sin t]}$

$y(0) = \boxed{C_1 = 0}$

$y'(t) = -2e^{-2t}(C_1 \cos t + C_2 \sin t) + e^{-2t}(-C_1 \sin t + C_2 \cos t)$

$y'(0) = -2C_1 + C_2 = 1 \Rightarrow C_2 = 1$

$\boxed{y(t) = e^{-2t} \sin t}$

11. Determine the longest interval in which the given initial value problem is certain to have a unique solution.

$$(x-2)y'' + y' + (x-2)(\tan x)y = 0, \qquad y(3) = 1, y'(3) = 2.$$

12. If the Wronskian of f and g is $3e^{4t}$ and $f(t) = e^{2t}$, find $g(t)$.

13. A spring is stretch 10 cm by a force of 3 N. A mass of 2 kg is hung from the spring and is also attached to a viscous damper that exerts a force of 3 N when the velocity of the mass 5 m/s. If the mass is pulled down 5 cm below its equilibrium position and given an initial velocity of 10 cm/s, determine its position u at any time. Find the quasifrequency of the motion.

cm → m (always!)

$3 = 0.1k \Rightarrow \boxed{k=30}$

$m = 2$

damping force: $3 = 5\gamma$

$\gamma = \frac{3}{5}$

$u(0) = 0.05$

$u'(0) = 0.1$

$mu'' + \gamma u' + ku = 0$

$2u'' + \frac{3}{5}u' + 30u = 0$

$10u'' + 3u' + 150u = 0$, $u(0) = 0.05$, $u'(0) = 0.1$

solve the initial value problem.

auxiliary equation: $10r^2 + 3r + 150 = 0$

$$r_1 = \frac{-3+\sqrt{9-4(10)(150)}}{20} = \frac{-3+\sqrt{-5991}}{20} = -\frac{3}{20} + \frac{i\sqrt{5991}}{20}$$

$r_2 = \overline{r_1}$

General solution:

$$u(t) = \left(C_1 \cos \frac{\sqrt{5991}}{20}t + C_2 \sin \frac{\sqrt{5991}}{20}t\right)e^{-\frac{3}{20}t}$$

$$\boxed{\text{quasifrequency} = \frac{\sqrt{5991}}{20}}$$

Plug $u(t)$ into the initial conditions: $u(0) = \boxed{C_1 = 0.05}$

$$u'(t) = \left(-C_1\frac{\sqrt{5991}}{20}\sin\frac{\sqrt{5991}}{20}t + C_2\frac{\sqrt{5991}}{20}\cos\frac{\sqrt{5991}}{20}t\right)e^{-\frac{3}{20}t}$$
$$-\frac{3}{20}\left(C_1\cos\frac{\sqrt{5991}}{20}t + C_2\sin\frac{\sqrt{5991}}{20}t\right)e^{-\frac{3}{20}t}$$

$$u'(0) = C_2\frac{\sqrt{5991}}{20} - C_1\frac{3}{20} = 0.1$$

$$C_2\sqrt{5991} - 3C_1 = 2 \Rightarrow \boxed{C_2 = \frac{2+3C_1}{\sqrt{5991}} = \frac{2.15}{\sqrt{5991}}}$$

$$\boxed{u(t) = \left(0.05\cos\frac{\sqrt{5991}}{20}t + \frac{2.15}{\sqrt{5991}}\sin\frac{\sqrt{5991}}{20}t\right)e^{-\frac{3}{20}t}}$$

$m = \frac{8}{32} = \frac{1}{4}$

14. A mass weighting 8 lb is attached to a spring hanging from the ceiling and comes to rest at its equilibrium position. At $t = 0$, an external force $F(t) = 2\cos 2t$ lb is applied to the system. If the spring constant is 10 lb/ft and the damping constant is 1 lb-sec/ft, find the steady-state solution for the system.

$$mu'' + \gamma u' + ku = 2\cos 2t$$

$$\frac{8}{32}u'' + u' + 10u = 2\cos 2t$$

$$u'' + 4u' + 40u = 8\cos 2t$$

steady-state solution $= u_p(t)$ particular solution

$$u_p(t) = A\cos 2t + B\sin 2t$$

$$u_p'(t) = -2A\sin 2t + 2B\cos 2t$$

$$u_p''(t) = -4A\cos 2t - 4B\sin 2t$$

$$-4A\cos 2t - 4B\sin 2t - 8A\sin 2t + 8B\cos 2t + 40A\cos 2t + 40B\sin 2t = 8\cos 2t$$

$\cos 2t$: $36A + 8B = 8$

$\sin 2t$: $36B - 8A = 0 \Rightarrow A = \frac{36B}{8} = \frac{9B}{2}$

$$36\left(\frac{9B}{2}\right) + 8B = 8$$

$$\frac{18(9B) + 8B = 8}{2 \qquad\qquad 2} \Longrightarrow 81B + 4B = 4$$

$$85B = 4 \Rightarrow \boxed{B = \frac{4}{85}}, \boxed{A = \frac{18}{85}}$$

steady-state solution: $\boxed{u(t) = \frac{18}{85}\cos 2t + \frac{4}{85}\sin 2t}$

resonance frequency: $\frac{\omega_r}{2\pi} = \frac{\sqrt{\frac{k}{m} - \frac{\gamma^2}{2m^2}}}{2\pi} = \ldots$

15. A mass weighing 4 lb stretches a spring 1.5 in. The mass is given a positive displacement 2 in from its equilibrium position and released with no initial velocity. Assuming that there is no damping and the mass is acted on by an external force of $2\cos 3t$ lb,

(a) Formulate the initial value problem describing the motion of mass

(b) Solve the initial value problem.

(c) If the given external force is replaced by a force $4\cos\omega t$ of frequency ω, find the value of ω for which resonance occurs.

$m = \frac{4}{32} = \frac{1}{8}$, $\quad 4 = k \cdot \frac{1.5}{12} \Rightarrow k = \frac{48}{1.5} = 32$, $\gamma = 0$

$\frac{1}{8}u'' + 32u = 2\cos 3t$

$u'' + 256u = 16\cos 3t$, $\quad u(0) = \frac{2}{12} = \frac{1}{6}$, $\quad u'(0) = 0$

transient part $\quad u_h(t) = C_1\cos 16t + C_2\sin 16t$

steady-state solution $\quad u_p(t) = A\cos 3t + B\sin 3t$

$B = 0$, $A = \frac{16}{247}$

$$\boxed{\begin{aligned} u(t) &= C_1\cos 16t + C_2\sin 16t + \frac{16}{247}\cos 3t \\ C_1 &= \frac{1}{6} - \frac{16}{247} \\ C_2 &= 0 \end{aligned}}$$

$F = 4\cos\omega t$, $\quad \boxed{\omega = 16}$

resonance occurs.

16. Find the general solution of the equation

(a) $y'' + 6y' + 9y = \frac{e^{-3x}}{1+2x}$

Variation of parameters.

$y'' + 6y' + 9y = 0, \quad r^2 + 6r + 9 = 0, \quad (r+3)^2 = 0, \quad r = -3$ - repeated root

$$y_h(x) = (C_1 + C_2 x)e^{-3x} = C_1 \underbrace{e^{-3x}}_{y_1(x)} + C_2 \underbrace{x e^{-3x}}_{y_2(x)}$$

$y_1(x) = e^{-3x}, \quad y_2(x) = xe^{-3x}$

$$W[y_1, y_2] = \begin{vmatrix} e^{-3x} & xe^{-3x} \\ -3e^{-3x} & e^{-3x} - 3xe^{-3x} \end{vmatrix} = e^{-3x}\left(e^{-3x} - \cancel{3xe^{-3x}}\right) + \cancel{3xe^{-3x} \cdot e^{-3x}} = e^{-6x}$$

$$c_1(x) = -\int \frac{y_2(x)g(x)}{W[y_1, y_2]}dx = -\int \frac{x\cancel{e^{-3x}}}{\cancel{e^{-6x}}} \cdot \frac{\cancel{e^{-3x}}}{1+2x}dx = -\int \frac{x\,dx}{1+2x} \quad \left| \begin{array}{l} u = 1+2x \\ x = \frac{u-1}{2} \\ dx = \frac{du}{2} \end{array} \right|$$

$$= -\int \frac{\frac{u-1}{2} \cdot \frac{du}{2}}{u} = -\frac{1}{4}\int (u-1)u^{-1}du = -\frac{1}{4}\int\left(1 - \frac{1}{u}\right)du$$

$$= -\frac{1}{4}\left(u - \ln|u|\right) + C_3 = -\frac{1}{4}\left(1+2x - \ln|1+2x|\right) + C_3$$

$$c_2(x) = \int \frac{y_1(x)g(x)}{W[y_1, y_2]}dx = \int \frac{\cancel{e^{-3x}}}{\cancel{e^{-6x}}} \cdot \frac{\cancel{e^{-3x}}}{1+2x}dx = \int \frac{dx}{1+2x} = \ln|1+2x| + C_4$$

General solution:

$$\boxed{y(x) = -\frac{1}{4}\left(1+2x - \ln|1+2x|\right)e^{-3x} + C_3 e^{-3x} + xe^{-3x}\ln|1+2x| + C_4 xe^{-3x}}$$

(b) $y'' + 2y' + y = 4e^{-t}$, $y(0) = 2$, $y'(0) = 1$

$y''+2y'+y=0 \Rightarrow r^2+2r+1=0 \Rightarrow (r+1)^2=0 \Rightarrow r=-1$ repeated root

$y_h(t) = (C_1 + C_2 t)e^{-t}$

$y_p(t) = At^2 e^{-t}$

$r=-1$ is the repeated root of the auxiliary equation.

$y_p' = 2At\,e^{-t} - At^2 e^{-t}$

$y_p'' = 2Ae^{-t} - 2At\,e^{-t} - 2At\,e^{-t} + At^2 e^{-t}$

$= 2Ae^{-t} - 4At\,e^{-t} + At^2 e^{-t}$

$2Ae^{-t} - \cancel{4Ate^{-t}} + \cancel{At^2e^{-t}} + \cancel{4Ate^{-t}} - \cancel{2At^2e^{-t}} + \cancel{At^2e^{-t}} = 4e^{-t}$

$2Ae^{-t} = 4e^{-t} \Rightarrow \boxed{A=2}$

$y_p = 2t^2 e^{-t}$

$\boxed{y(t) = (C_1 + C_2 t)e^{-t} + 2t^2 e^{-t}}$

(c) $y'' + 4y = 32\sin 2t - 32t\cos 2t$

17. For the equation $y'' + xy' + 2y = 0$

(a) Seek its power series solution about $x_0 = 0$; find the recurrence relation.

[redacted]

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

$$\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} + x \sum_{n=1}^{\infty} a_n n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} + \sum_{n=1}^{\infty} a_n n \overbrace{x x^{n-1}}^{x^n} + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$n-2 \longrightarrow n$

$$\sum_{n=0}^{\infty} a_{n+2} (n+2)(n+1) x^n + \sum_{n=1}^{\infty} a_n n x^n + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$a_{0+2}(0+2)(0+1) x^0 + \sum_{n=1}^{\infty} a_{n+2}(n+2)(n+1) x^n + \sum_{n=1}^{\infty} a_n n x^n + 2a_0 x^0 + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty} \left[a_{n+2}(n+2)(n+1) + a_n n + 2a_n \right] x^n = 0$$

Recurrence relation:

$$2a_2 + 2a_0 = 0 \Rightarrow \boxed{a_2 = -a_0}$$

$$a_{n+2}(n+2)(n+1) + a_n n + 2a_n = 0$$

$$a_{n+2} (\cancel{n+2})(n+1) = -(\cancel{n+2}) a_n$$

$$\boxed{a_{n+2} = -\frac{a_n}{n+1}}$$ recurrence relation.

18. Determine a lower bound for the radius of convergence of series solution for the equation

$$(x^2 + x)y'' + 3y' - 6xy = 0$$

singular points

$x^2 + x = 0$

$x(x+1) = 0$

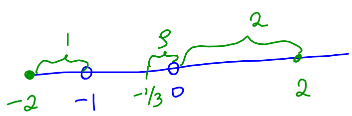
$x_1 = 0, \quad x_2 = -1$

about

(a) $x_0 = -2$, $\rho = 1$

(b) $x_0 = -\dfrac{1}{3}$, $\rho = 1/3$

(c) $x_0 = 2$, $\rho = 2$



19. Determine $y'''(1)$ if $y(x)$ is the solution of the initial value problem

$$x^2y'' + (1+x)y' + 3(\ln x)y = 0, \qquad y(1) = 2, y'(1) = 0$$

(a) $f(t) = \begin{cases} \frac{t}{2}, & 0 \le t < 6 \\ 3, & t \ge 6 \end{cases}$

1. Find the Laplace transform of the given function.

(a) $f(t) = \begin{cases} \frac{t}{2}, & 0 \le t < 6 \\ 3, & t \ge 6 \end{cases}$, $f(t) = \frac{t}{2} + \left(3 - \frac{t}{2}\right) u_6(t)$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{\frac{t}{2} + \left(3 - \frac{t}{2}\right) u_6(t)\right\}$$

$$= \mathcal{L}\left\{\frac{t}{2}\right\} + \mathcal{L}\left\{\frac{1}{2}(6 - t) u_6(t)\right\}$$

$$= \frac{1}{2s^2} + \frac{1}{2}(-1)\, \mathcal{L}\{(t-6) u_6(t)\}$$

$$= \frac{1}{2s^2} - \frac{1}{2} e^{-6s} \mathcal{L}\{t\} = \boxed{\frac{1}{2s^2} - \frac{e^{-6s}}{2s^2}}$$

(b) $f(t) = (t^2 - 2t + 2)u_1(t)$

$$f(t) = [(t-1)^2 + 1]\, u_1(t)$$

$$= (t-1)^2 u_1(t) + u_1(t)$$

$$\mathcal{L}\{(t-1)^2 u_1(t) + u_1(t)\}$$

$$= e^{-s} \mathcal{L}\{t^2\} + \frac{e^{-s}}{s}$$

$$= \boxed{e^{-s} \frac{2}{s^3} + \frac{e^{-s}}{s}}$$

(b) $f(t) = (t^2 - 2t + 2)u_1(t) = [(t-1)^2+1]u_1(t) = (t-1)^2 u_1(t) + u_1(t)$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{(t-1)^2 u_1(t)\} + \mathcal{L}\{u_1(t)\} = \mathcal{L}\{t^2\} e^{-cs} + \dfrac{e^{-cs}}{s}$

$= \boxed{\dfrac{2}{s^3} e^{-cs} + \dfrac{e^{-cs}}{s}}$

(c) $f(t) = \int_0^t (t-\tau)^2 \cos 2\tau \, d\tau = (g * h)(t)$

$g(t-\tau) = (t-\tau)^2 \Rightarrow g(t) = t^2$

$h(\tau) = \cos 2\tau \Rightarrow h(t) = \cos 2t$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\} \cdot \mathcal{L}\{h(t)\}$

$= \mathcal{L}\{t^2\} \cdot \mathcal{L}\{\cos 2t\}$

$= \boxed{\dfrac{2}{s^3} \cdot \dfrac{s}{s^2+4}}$

(d) $f(t) = t \cos 3t$

$\mathcal{L}\{t \cos 3t\} = (-1) \dfrac{d}{ds}\{\mathcal{L}\{\cos 3t\}\}$

$= -\dfrac{d}{ds}\left(\dfrac{s}{s^2+9}\right)$

$= -\dfrac{s^2+9-2s(s)}{(s^2+9)^2}$

$= \boxed{\dfrac{s^2-9}{(s^2+9)^2}}$

(e) $f(t) = e^t \delta(t-1)$

$\mathcal{L}\{\delta(t-1)\} = e^{-s}$

$\mathcal{L}\{e^t \delta(t-1)\} = \boxed{e^{-(s-1)}}$

21. Find the inverse Laplace transform of the given function.

(a) $F(s) = \dfrac{2s+6}{s^2-4s+8}$

2. Find the inverse Laplace transform of the given function.

(a) $F(s) = \dfrac{2s+6}{s^2-4s+8}$

$$\frac{2s+6}{s^2-4s+8} = \frac{2s+6}{(s-2)^2+4} = 2\,\frac{s+3}{(s-2)^2+4}$$

$$= 2\,\frac{s-2+5}{(s-2)^2+4} = 2\,\frac{s-2}{(s-2)^2+4} + 5\cdot\frac{2}{(s-2)^2+4}$$

$$\mathcal{L}^{-1}\{F(s)\} = 2\,\mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+4}\right\} + 5\,\mathcal{L}^{-1}\left\{\frac{2}{(s-2)^2+4}\right\}$$

$$= \boxed{2e^{2t}\cos 2t + 5e^{2t}\sin 2t}$$

(b) $F(s) = \dfrac{e^{-2s}}{s^2+s-2}$

$$\frac{1}{s^2+s-2} = \frac{1}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1}$$

$$= \frac{A(s-1)+B(s+2)}{(s+2)(s-1)}$$

$$1 = A(s-1)+B(s+2)$$

$s=1$: $\quad 1 = 3B \Rightarrow B = 1/3$

$s=-2$: $\quad 1 = -3A \Rightarrow A = -1/3$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+s-2}\right\} = \mathcal{L}\left\{-\frac{1}{3}\,\frac{1}{s+2} + \frac{1}{3}\,\frac{1}{s-1}\right\}$$

$$= -\frac{1}{3}e^{-2t} + \frac{1}{3}e^{t}$$

$$\mathcal{L}\left\{\frac{e^{-2s}}{s^2+s-2}\right\} = \boxed{\left[-\frac{1}{3}e^{-2(t-2)} + \frac{1}{3}e^{(t-2)}\right]u_2(t)}$$

22. Solve the initial value problem using the Laplace transform:

(a) $\mathcal{L}\{y'' + 4y\} = \mathcal{L}\left\{\begin{cases} t, & 0 \le t < 1 \\ 1, & t \ge 1 \end{cases}\right\}$, $y(0) = y'(0) = 0$

$\mathcal{L}\{f(t-c)\,u_c(t)\} = e^{-cs}\mathcal{L}\{f\}$

$$\begin{cases} t, & 0 \le t < 1 \\ 1, & t \ge 1 \end{cases} = t + (1-t)u_1(t) = t - (t-1)\,u_1(t)$$

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{t - (t-1)u_1(t)\} = \mathcal{L}\{t\} - \mathcal{L}\{(t-1)u_1(t)\} = \frac{1}{s^2} - e^{-s}\mathcal{L}\{t\} = \frac{1}{s^2} - \frac{e^{-s}}{s^2}$$

$$\mathcal{L}\{y\} = Y(s), \quad \mathcal{L}\{y''\} = s^2Y(s) - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0} = s^2Y(s)$$

$$s^2Y(s) + 4Y(s) = \frac{1}{s^2} - \frac{e^{-s}}{s^2}$$

$$(s^2+4)\,Y(s) = \frac{1}{s^2} - \frac{e^{-s}}{s^2}$$

$$Y(s) = \frac{1}{s^2(s^2+4)} - \frac{e^{-s}}{s^2(s^2+4)}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+4)} - \frac{e^{-s}}{s^2(s^2+4)}\right\}$$

Partial fractions: $\dfrac{1}{s^2(s^2+4)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{Cs+D}{s^2+4}$, $\quad A = C = 0$, $B = 1/4$, $D = -1/4$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{4}\,\frac{1}{s^2} - \frac{1}{4\cdot 2}\,\frac{2}{s^2+4}\right\}$$

$$= \frac{1}{4}t - \frac{1}{8}\sin 2t$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2(s^2+4)}\right\} = u_1(t)\left[\frac{1}{4}(t-1) - \frac{1}{8}\sin 2(t-1)\right]$$

$\mathcal{L}^{-1}\{e^{-cs}F(s)\} = f(t-c)\,u_c(t)$

$$\boxed{y(t) = \frac{1}{4}t - \frac{1}{8}\sin 2t + u_1(t)\left[\frac{1}{4}(t-1) - \frac{1}{8}\sin 2(t-1)\right]}$$

(b) $y'' + 2y' + 3y = \delta(t - 3\pi)$, $y(0) = y'(0) = 0$

(b) $y'' + 2y' + 3y = \delta(t - 3\pi)$, $y(0) = y'(0) = 0$

$\mathcal{L}\{y'' + 2y' + 3y\} = \mathcal{L}\{\delta(t-3\pi)\}$

$\mathcal{L}\{y\} = Y(s)$

$\mathcal{L}\{y'\} = sY(s) - y(0)$

$= sY(s)$

$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0)$

$= s^2 Y(s)$

$\mathcal{L}\{\delta(t-3\pi)\} = e^{-3\pi s}$

$(s^2+2s+3)\,Y(s) = e^{-3\pi s}$

$Y(s) = \dfrac{1}{s^2+2s+3}\, e^{-3\pi s}$

$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\dfrac{e^{-3\pi s}}{s^2+2s+3}\right\}$

$\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+2s+3}\right\}$

$= \mathcal{L}^{-1}\left\{\dfrac{1}{(s+1)^2+2}\right\}$

$= \dfrac{1}{\sqrt{2}}\mathcal{L}^{-1}\left\{\dfrac{\sqrt{2}}{(s+1)^2+2}\right\}$

$= \dfrac{1}{\sqrt{2}} e^{-t} \sin\sqrt{2}\, t$

$\mathcal{L}^{-1}\left\{\dfrac{e^{-3\pi s}}{s^2+2s+3}\right\}$

$= \boxed{u_{3\pi}(t)\, \dfrac{1}{\sqrt{2}}\, e^{-(t-3\pi)} \sin\sqrt{2}(t-3\pi) = y(t)}$

(c) $y'' + 4y' + 4y = g(t)$, $y(0) = 2$, $y'(0) = -3$

(c) $y'' + 4y' + 4y = g(t)$, $y(0) = 2$, $y'(0) = -3$

$$\mathcal{L}\{y'' + 4y' + 4y\} = \mathcal{L}\{g(t)\}$$

$$\mathcal{L}\{g(t)\} = G(s)$$

$$\mathcal{L}\{y\} = Y(s)$$

$$\mathcal{L}\{y'\} = sY(s) - y(0) = sY(s) - 2$$

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0)$$

$$= s^2Y(s) - 2s + 3$$

$$s^2Y(s) - 2s + 3 + 4sY(s) - 8 + 4Y(s) = G(s)$$

$$Y(s)(s^2+4s+4) = G(s) + 2s + 5$$

$$Y(s) = \frac{G(s)}{s^2+4s+4} + \frac{2s+5}{s^2+4s+4}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{G(s)}{s^2+4s+4} + \frac{2s+5}{s^2+4s+4}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\}$$

$$= e^{-2t}t$$

$$\frac{2s+5}{s^2+4s+4} = \frac{A}{s+2} + \frac{B}{(s+2)^2}$$

$$= \frac{A(s+2) + B}{(s+2)^2}$$

$$2s+5 = A(s+2) + B$$

$s=-2$: $1 = B$

$s=0$: $5 = 2A + B$, $2A = 5 - B = 4$

$A = 2$

$$\frac{2s+5}{s^2+4s+4} = \frac{2}{s+2} + \frac{1}{(s+2)^2}$$

$$\mathcal{L}^{-1}\left\{\frac{2s+5}{s^2+4s+4}\right\} = 2e^{-2t} + e^{-2t}t$$

$$\boxed{y(t) = \int_0^t g(t-\tau)e^{-2\tau}\tau\,d\tau + 2e^{-2t} + e^{-2t}t}$$

10. Find A^{-1} if $A = \begin{pmatrix} 1+i & -1+2i \\ 3+2i & 2-i \end{pmatrix}$

$$\det A = (1+i)(2-i) - (-1+2i)(3+2i) = 2 - i + 2i - \overbrace{i^2}^{-1} + 3 + 2i - 6i - 4\overbrace{i^2}^{-1}$$

$$= 5 + 5 - 3i = 10 - 3i$$

$$\frac{1}{\det A} = \frac{1}{10-3i} = \frac{10+3i}{(10-3i)(10+3i)} = \frac{10+3i}{100-9i^2} = \frac{10+3i}{109}$$

$$\boxed{A^{-1} = \frac{10+3i}{109}\begin{pmatrix} 2-i & 1-2i \\ -3-2i & 1+i \end{pmatrix}}$$

5. Find BA if $A = \begin{pmatrix} 1+i & -1+2i \\ 3+2i & 2-i \end{pmatrix}$, $B = \begin{pmatrix} i & 3 \\ 2 & -2i \end{pmatrix}$

$$BA = \begin{pmatrix} i & 3 \\ 2 & -2i \end{pmatrix}\begin{pmatrix} 1+i & -1+2i \\ 3+2i & 2-i \end{pmatrix} = \begin{pmatrix} i(1+i)+3(3+2i) & i(-1+2i)+3(2-i) \\ 2(1+i)-2i(3+2i) & 2(-1+2i)-2i(2-i) \end{pmatrix}$$

$$= \begin{pmatrix} i+i^2+9+6i & -i+2i^2+6-3i \\ 2+2i-6i-4i^2 & -2+4i-4i+4i^2 \end{pmatrix} = \boxed{\begin{pmatrix} 8+7i & 4-4i \\ 6-4i & -6 \end{pmatrix}}$$

25. Find the general solution of the system. Classify the critical point (0,0) as to type, determine whether it is stable or unstable, sketch the phase portrait.

(a) $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x}$, $A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$, $tr(A) = 1-2=-1$
$det(A) = -2-4=-6$.

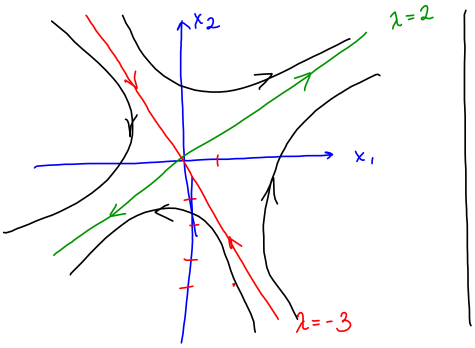
characteristic equation $\lambda^2+\lambda-6=0$
$(\lambda+3)(\lambda-2)=0$

$\boxed{\lambda_1=-3, \quad \vec{v}=\begin{pmatrix} 1 \\ -4 \end{pmatrix}}$ see 26(a)

$\boxed{\lambda_2=2, \quad \vec{v}=\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

General solution $\boxed{\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}}$ $\Psi = \begin{pmatrix} 1e^{-3t} & 1e^{2t} \\ -4e^{-3t} & 1e^{2t} \end{pmatrix}$

Phase portrait :



(0,0) is a saddle point
unstable.

(b) $\mathbf{x}' = \begin{pmatrix} -3 & -1 \\ \boxed{1} & -1 \end{pmatrix} \mathbf{x}$ CCW , $A = \begin{pmatrix} -3 & -1 \\ 1 & -1 \end{pmatrix}$, $tr(A) = -3-1 = -4$

$\det A = 3+1 = 4$

characteristic equation $\lambda^2 + 4\lambda + 4 = 0$

$(\lambda+2)^2 = 0 \Rightarrow \boxed{\lambda = -2}$, repeated

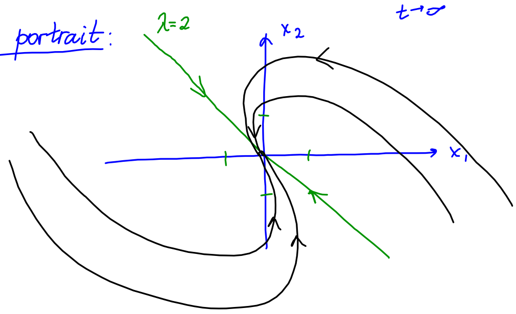
eigenvector $\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

generalized eigenvector $\vec{w} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

General solution $\vec{x}(t) = \left[c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \left(t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right) \right] e^{-2t}$

$\lim_{t \to \infty} \vec{x}(t) = \vec{0}$

Phase portrait:



(0,0) is an improper node

asymptotically stable

(c) $\mathbf{x}' = \begin{pmatrix} -3 & 2 \\ \underset{\text{cw}}{\boxed{-1}} & -1 \end{pmatrix}\mathbf{x}$, $A = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix}$, $tr(A) = -3-1 = -4$
$det(A) = 3+2 = 5$

<u>Characteristic equation</u> $\lambda^2 + 4\lambda + 5 = 0$

<u>eigenvalues</u> $\lambda = -2 \pm i$

<u>eigenvector</u> $\lambda = -2+i$, $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $\left(A - (-2+i)I\right)\vec{v} = \vec{0}$

$$\begin{pmatrix} -3-(-2+i) & 2 \\ -1 & -1-(-2+i) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1-i & 2 \\ -1 & 1-i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} (-1-i)\,v_1 + 2v_2 = 0 \\ -v_1 + (1-i)\,v_2 = 0 \end{cases}$$
$$v_1 = (1-i)\,v_2$$

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} (1-i)v_2 \\ v_2 \end{pmatrix} \overset{v_2=1}{=} \boxed{\begin{pmatrix} 1-i \\ 1 \end{pmatrix}, \quad \lambda = -2+i}$$

$$\vec{v}e^{\lambda t} = \begin{pmatrix} 1-i \\ 1 \end{pmatrix} e^{(-2+i)t} = \begin{pmatrix} 1-i \\ 1 \end{pmatrix} \overbrace{e^{-2t}\left(\cos t + i\sin t\right)}^{e^{(-2+i)t}}$$

$$= e^{-2t}\begin{pmatrix} (1-i)(\cos t + i\sin t) \\ \cos t + i\sin t \end{pmatrix} = e^{-2t}\begin{pmatrix} \cos t + i\sin t - i\cos t - \overset{-1}{\cancel{i^2}}\sin t \\ \cos t + i\sin t \end{pmatrix}$$

$$= e^{-2t}\left(\begin{pmatrix} (\cos t + \sin t) \\ \cos t \end{pmatrix} + i\begin{pmatrix} (\sin t - \cos t) \\ \sin t \end{pmatrix}\right)$$

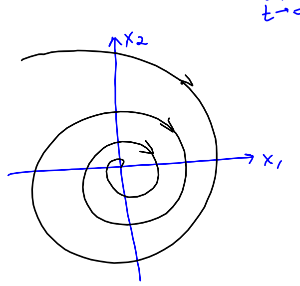
$$= e^{-2t}\left[\begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + i\begin{pmatrix} \sin t - \cos t \\ \sin t \end{pmatrix}\right]$$

<u>General solution</u> :

$$\vec{x}(t) = e^{-2t}\left[c_1\begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + c_2\begin{pmatrix} \sin t - \cos t \\ \sin t \end{pmatrix}\right]$$

$$\lim_{t\to\infty} \vec{x}(t) = 0$$

<u>Phase portrait</u>



(0,0) is a <u>spiral sink</u>
<u>asymptotically stable</u>

$\lambda^2 - (tr\, A)\lambda + \det A = 0$

26. Find the general solution of the system using variation of parameters and Laplace Transform, if possible.

(a) $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}$

Undetermined coefficients.

Corresponding homogeneous system: $\vec{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\vec{x}$, $A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$, $tr(A) = 1-2 = -1$, $\det(A) = -2-4 = -6$

Characteristic equation $\lambda^2 + \lambda - 6 = 0$
$(\lambda+3)(\lambda-2) = 0$ or $\boxed{\lambda_1 = -3,\ \lambda_2 = 2}$ eigenvalues.

Corresponding eigenvectors: $\lambda_1 = -3$, $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, is a solution of $(A - (-3)I)\vec{v} = \vec{0}$
$(A + 3I)\vec{v} = \vec{0}$

$$\begin{pmatrix} 1+3 & 1 \\ 4 & -2+3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ or } \begin{matrix} 4v_1 + v_2 = 0 \\ v_2 = -4v_1 \end{matrix}$$

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ -4v_1 \end{pmatrix} \underset{=}{\overset{v_1=1}{}} \boxed{\begin{pmatrix} 1 \\ -4 \end{pmatrix},\ \lambda = -3}$$

$\lambda_2 = 2$. $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$, is a solution of $(A - 2I)\vec{w} = \vec{0}$

$$\begin{pmatrix} 1-2 & 1 \\ 4 & -2-2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ or } \begin{matrix} -w_1 + w_2 = 0 \\ w_1 = w_2 \end{matrix}$$

$$\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_1 \end{pmatrix} \underset{=}{\overset{w_1=1}{}} \boxed{\begin{pmatrix} 1 \\ 1 \end{pmatrix},\ \lambda = 2}$$

$$\boxed{\vec{x}_h(t) = c_1\begin{pmatrix} 1 \\ -4 \end{pmatrix}e^{-3t} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{2t}}$$ general solution of the homogeneous system

$\vec{x}_p(t) = \vec{a}\, e^{-2t} + \vec{b}\, e^{t}$, $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

$\vec{x}_p'(t) = -2\vec{a}e^{-2t} + \vec{b}e^{t}$

$$\vec{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \vec{x} + \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}$$

$$\vec{x}_p(t) = \vec{a}\, e^{-2t} + \vec{b}\, e^{t}, \quad \vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$\vec{x}_p'(t) = -2\vec{a} e^{-2t} + \vec{b} e^{t}$$

$$-2\vec{a} e^{-2t} + \vec{b} e^{t} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \left(\vec{a} e^{-2t} + \vec{b} e^{t} \right) + \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}$$

$$-2\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{-2t} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^{t} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \left[\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{-2t} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^{t} \right] + \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}$$

$$\begin{pmatrix} -2a_1 e^{-2t} + b_1 e^{t} \\ -2a_2 e^{-2t} + b_2 e^{t} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} a_1 e^{-2t} + b_1 e^{t} \\ a_2 e^{-2t} + b_2 e^{t} \end{pmatrix} + \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}$$

$$\begin{pmatrix} -2a_1 e^{-2t} + b_1 e^{t} \\ -2a_2 e^{-2t} + b_2 e^{t} \end{pmatrix} = \begin{pmatrix} a_1 e^{-2t} + b_1 e^{t} + a_2 e^{-2t} + b_2 e^{t} + e^{-2t} \\ 4(a_1 e^{-2t} + b_1 e^{t}) - 2(a_2 e^{-2t} + b_2 e^{t}) - 2e^{t} \end{pmatrix}$$

$$\begin{pmatrix} -2a_1 e^{-2t} + b_1 e^{t} \\ -2a_2 e^{-2t} + b_2 e^{t} \end{pmatrix} = \begin{pmatrix} e^{-2t}(a_1 + a_2 + 1) + e^{t}(b_1 + b_2) \\ e^{-2t}(4a_1 - 2a_2) + e^{t}(4b_1 - 2b_2 - 2) \end{pmatrix}$$

1st component

e^{-2t}: $\left\{ -2a_1 = a_1 + a_2 + 1 \Rightarrow a_2 + 1 = 0 \quad \boxed{a_2 = -1} \right.$

e^{t}: $\left\{ b_1 = b_1 + b_2 \Rightarrow \boxed{b_2 = 0} \right.$

2nd component

e^{-2t}: $\left\{ -2a_2 = 4a_1 - 2a_2 \Rightarrow 4a_1 = 0 \Rightarrow a_1 = 0. \right.$

e^{t}: $\left\{ b_2 = 4b_1 - 2b_2 - 2 \Rightarrow 4b_1 - 2 = 0 \text{ or } \boxed{b_1 = 1/2} \right.$

$$\vec{x}_p(t) = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{-2t} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^{t}$$

$$= \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{-2t} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} e^{t}$$

$$\boxed{\vec{x}_p(t) = \begin{pmatrix} \frac{1}{2} e^{t} \\ -e^{-2t} \end{pmatrix}}$$ particular solution of the nonhomogeneous system.

$$\boxed{\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t) = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} \frac{1}{2} e^{t} \\ -e^{-2t} \end{pmatrix}}$$

Laplace Transform $\quad \vec{x}(0)=\vec{0}$

$$\mathcal{L}\{\vec{x}(t)\} = \vec{X}(s), \quad \mathcal{L}\{\vec{x}\,'(t)\} = s\vec{X}(s) - \cancel{\vec{x}(0)}^{\,0} = s\,\vec{x}(s)$$

$$\mathcal{L}\left\{\begin{pmatrix} e^{-2t} \\ -2e^{t}\end{pmatrix}\right\} = \begin{pmatrix}\frac{1}{s+2}\\ -\frac{2}{s-1}\end{pmatrix}$$

$$s\vec{X}(s) = \begin{pmatrix}1&1\\4&-2\end{pmatrix}\vec{X}(s) + \begin{pmatrix}\frac{1}{s+2}\\ -\frac{2}{s-1}\end{pmatrix}$$

$$sI\vec{X}(s) - \begin{pmatrix}1&1\\4&-2\end{pmatrix}\vec{X}(s) = \begin{pmatrix}\frac{1}{s+2}\\ -\frac{2}{s-1}\end{pmatrix}$$

$$\left[sI - \begin{pmatrix}1&1\\4&-2\end{pmatrix}\right]\vec{X}(s) = \begin{pmatrix}\frac{1}{s+2}\\ -\frac{2}{s-1}\end{pmatrix}$$

$$\left[\begin{pmatrix}s&0\\0&s\end{pmatrix} - \begin{pmatrix}1&1\\4&-2\end{pmatrix}\right]\vec{X}(s) = \begin{pmatrix}\frac{1}{s+2}\\ -\frac{2}{s-1}\end{pmatrix}$$

$$\begin{pmatrix}s-1&-1\\-4&s+2\end{pmatrix}\vec{X}(s) = \begin{pmatrix}\frac{1}{s+2}\\ -\frac{2}{s-1}\end{pmatrix} \Longrightarrow \vec{X}(s) = \begin{pmatrix}s-1&-1\\-4&s+2\end{pmatrix}^{-1}\begin{pmatrix}\frac{1}{s+2}\\ -\frac{2}{s-1}\end{pmatrix}$$

$$\begin{vmatrix}s-1&-1\\-4&s+2\end{vmatrix} = (s-1)(s+2) - 4 = s^2+s-2-4 = s^2+s-6$$

$$\begin{pmatrix}s-1&-1\\-4&s+2\end{pmatrix}^{-1} = \frac{1}{s^2+s-6}\begin{pmatrix}s+2&1\\4&s-1\end{pmatrix} = \begin{pmatrix}\frac{s+2}{s^2+s-6}&\frac{1}{s^2+s-6}\\ \frac{4}{s^2+s-6}&\frac{s-1}{s^2+s-6}\end{pmatrix}$$

$$\vec{x}(s) = \begin{pmatrix}\frac{s+2}{s^2+s-6}&\frac{1}{s^2+s-6}\\ \frac{4}{s^2+s-6}&\frac{s-1}{s^2+s-6}\end{pmatrix}\begin{pmatrix}\frac{1}{s+2}\\ -\frac{2}{s-1}\end{pmatrix} = \begin{pmatrix}\frac{1}{s^2+s-6} - \frac{2}{(s-1)(s^2+s-6)}\\ \frac{4}{(s+2)(s^2+s-6)} - \frac{2}{s^2+s-6}\end{pmatrix}$$

$$\vec{x}(t) = \mathcal{L}^{-1}\{\vec{X}(s)\}$$

$$\vec{x}(s)=\begin{pmatrix}\frac{s+2}{s^2+s-6} & \frac{1}{s^2+s-6}\\ \frac{4}{s^2+s-6} & \frac{s-1}{s^2+s-6}\end{pmatrix}\begin{pmatrix}\frac{1}{s+2}\\ -\frac{2}{s-1}\end{pmatrix}=\begin{pmatrix}\frac{1}{s^2+s-6} - \frac{2}{(s-1)(s^2+s-6)}\\ \frac{4}{(s+2)(s^2+s-6)} - \frac{2}{s^2+s-6}\end{pmatrix}$$

$$\vec{x}(t)=\mathcal{L}^{-1}\{\vec{X}(s)\}$$

Partial fractions:

$$\frac{1}{s^2+s-6}=\frac{1}{(s+3)(s-2)}=\frac{A}{s+3}+\frac{B}{s-2}=\frac{1}{5}\left(\frac{1}{s-2}-\frac{1}{s+3}\right)$$

$$1=A(s-2)+B(s+3)$$

$s=2$: $1=5B \Rightarrow \boxed{B=1/5}$

$s=-3$: $1=-5A \Rightarrow \boxed{A=-1/5}$

$$\mathcal{L}^{-1}\left\{\frac{1}{5}\left(\frac{1}{s-2}-\frac{1}{s+3}\right)\right\}=\frac{1}{5}\left(e^{2t}-e^{-3t}\right)$$

$$-\frac{2}{(s-1)(s^2+s-6)}=-\frac{2}{(s-1)(s+3)(s-2)}=\frac{A}{s-1}+\frac{B}{s+3}+\frac{C}{s-2}=\frac{1}{2}\frac{1}{s-1}-\frac{1}{10}\frac{1}{s+3}-\frac{2}{5}\frac{1}{s-2}$$

$$-2=A(s+3)(s-2)+B(s-1)(s-2)+C(s-1)(s+3)$$

$s=-3$: $-2=20B \Rightarrow \boxed{B=-\frac{1}{10}}$

$s=2$: $-2=5C \Rightarrow \boxed{C=-\frac{2}{5}}$

$s=1$: $-2=-4A \Rightarrow \boxed{A=1/2}$

$$\mathcal{L}^{-1}\left\{-\frac{2}{(s-1)(s+3)(s-2)}\right\}=\frac{1}{2}e^{t}-\frac{1}{10}e^{-3t}-\frac{2}{5}e^{2t}$$

$$\frac{4}{(s+2)(s^2+s-6)}=\frac{4}{(s+2)(s+3)(s-2)}=\frac{A}{s+2}+\frac{B}{s+3}+\frac{C}{s-2}=-\frac{1}{s+2}+\frac{4}{5}\frac{1}{s+3}+\frac{1}{5}\frac{1}{s-2}$$

$$4=A(s+3)(s-2)+B(s+2)(s-2)+C(s+2)(s+3)$$

$s=2$: $4=20C \Rightarrow \boxed{C=\frac{4}{20}=\frac{1}{5}}$

$s=-3$: $4=5B \Rightarrow \boxed{B=\frac{4}{5}}$

$s=-2$: $4=-4A \Rightarrow \boxed{A=-1}$

$$\mathcal{L}^{-1}\left\{\frac{4}{(s+2)(s+3)(s-2)}\right\}=-e^{-2t}+\frac{4}{5}e^{-3t}+\frac{1}{5}e^{2t}$$

$$\boxed{\vec{x}(t)=\mathcal{L}^{-1}\{\vec{X}(s)\}=\begin{pmatrix}\frac{1}{5}\left(e^{2t}-e^{-3t}\right)+\frac{1}{2}e^{t}-\frac{1}{10}e^{-3t}-\frac{2}{5}e^{2t}\\ -e^{-2t}+\frac{4}{5}e^{-3t}+\frac{1}{5}e^{2t}-\frac{2}{5}\left(e^{2t}-e^{-3t}\right)\end{pmatrix}}$$

(b) $\mathbf{x}' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}\mathbf{x} + \begin{pmatrix} t^{-3} \\ -t^{-2} \end{pmatrix}$

Homogeneous system: $\vec{x}' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}\vec{x}$, $A = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}$, $tr(A) = 0$, $det(A) = 0$

Characteristic equation: $\lambda^2 = 0$

Eigenvalues $\lambda = 0$ (repeated).

corresponding eigenvector $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is a solution of $(A - 0 \cdot I)\vec{v} = \vec{0}$

$A\vec{v} = \vec{0}$

$$\begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or} \quad 4v_1 - 2v_2 = 0 \quad \text{or} \quad 2v_1 - v_2 = 0 \quad \text{or} \quad v_2 = 2v_1$$

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ 2v_1 \end{pmatrix} \underset{=}{\overset{v_1 = 1}{}} \boxed{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \lambda = 0}$$

Generalized eigenvector $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$, is a solution $(A - 0 \cdot I)\vec{w} = \vec{v}$

$$\begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow 4w_1 - 2w_2 = 1 \quad \text{or} \quad w_2 = \frac{4w_1 - 1}{2}$$

$$\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ \frac{4w_1 - 1}{2} \end{pmatrix} \underset{=}{\overset{w_1 = 0}{}} \begin{pmatrix} 0 \\ -1/2 \end{pmatrix}$$

$$\vec{x}_h(t) = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{0 \cdot t} + c_2\left[t\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ -1/2 \end{pmatrix}\right]e^{0 \cdot t}$$

$$\boxed{\vec{x}_h(t) = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} t \\ 2t - 1/2 \end{pmatrix}}$$ – general solution of the homogeneous system

Fundamental matrix: $\Psi(t)=\begin{pmatrix}1 & t\\ 2 & 2t-1/2\end{pmatrix}$, $\vec{g}(t)=\begin{pmatrix}1/t^3\\ -1/t^2\end{pmatrix}$

$$A=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$$

$$\boxed{A^{-1}=\frac{1}{\det A}\begin{pmatrix}a_{22} & -a_{12}\\ -a_{21} & a_{11}\end{pmatrix}}$$

Particular solution of the nonhomogeneous system

$$\vec{x}_p(t)=\Psi(t)\int\Psi^{-1}(t)\vec{g}(t)\,dt$$

$$\det\Psi(t)=\begin{vmatrix}1 & t\\ 2 & 2t-1/2\end{vmatrix}=2t-1/2-2t=-1/2$$

$$\Psi^{-1}(t)=\frac{1}{\det\Psi(t)}\begin{pmatrix}2t-1/2 & -t\\ -2 & 1\end{pmatrix}=-2\begin{pmatrix}2t-1/2 & -t\\ -2 & 1\end{pmatrix}=\begin{pmatrix}-4t+1 & 2t\\ 4 & -2\end{pmatrix}$$

$$\Psi^{-1}(t)\vec{g}(t)=\begin{pmatrix}-4t+1 & 2t\\ 4 & -2\end{pmatrix}\begin{pmatrix}\frac{1}{t^3}\\ -\frac{1}{t^2}\end{pmatrix}=\begin{pmatrix}(-4t+1)\frac{1}{t^3}-2t\frac{1}{t^2}\\ \frac{4}{t^3}+\frac{2}{t^2}\end{pmatrix}=\begin{pmatrix}-\frac{4}{t^2}+\frac{1}{t^3}-\frac{2}{t}\\ \frac{4}{t^3}+\frac{2}{t^2}\end{pmatrix}$$

$$\int\Psi^{-1}(t)g(t)\,dt=\begin{pmatrix}\int\left(-\frac{4}{t^2}+\frac{1}{t^3}-\frac{2}{t}\right)dt\\ \int\left(\frac{4}{t^3}+\frac{2}{t^2}\right)dt\end{pmatrix}=\begin{pmatrix}\frac{4}{t}-\frac{1}{2t^2}-2\ln|t|+\cancel{c_1}^{\,0}\\ -\frac{4}{2t^2}-\frac{2}{t}+\cancel{c_2}^{\,0}\end{pmatrix}$$

$$\vec{x}_p(t)=\Psi(t)\int\Psi^{-1}(t)g(t)\,dt=\begin{pmatrix}1 & t\\ 2 & 2t-1/2\end{pmatrix}\begin{pmatrix}\frac{4}{t}-\frac{1}{2t^2}-2\ln|t|\\ -\frac{4}{2t^2}-\frac{2}{t}\end{pmatrix}=\begin{pmatrix}\frac{4}{t}-\frac{1}{2t^2}-2\ln|t|+t\left(-\frac{2}{t^2}-\frac{2}{t}\right)\\ 2\left(\frac{4}{t}-\frac{1}{2t^2}-2\ln|t|\right)+(2t-1/2)\left(-\frac{2}{t^2}-\frac{2}{t}\right)\end{pmatrix}$$

$$=\begin{pmatrix}\frac{2}{t}-\frac{1}{2t^2}-2\ln|t|-2\\ \frac{8}{t}-\cancel{\frac{1}{t^2}}-4\ln|t|-\frac{4}{t}-4+\cancel{\frac{1}{t^2}}+\frac{1}{t}\end{pmatrix}$$

$$\boxed{\vec{x}_p(t)=\begin{pmatrix}\frac{2}{t}-\frac{1}{2t^2}-2\ln|t|-2\\ \frac{5}{t}-4\ln|t|-4\end{pmatrix}}$$

General solution of the nonhomogeneous system

$$\boxed{\vec{x}(t)=\vec{x}_h+\vec{x}_p=c_1\begin{pmatrix}1\\ 2\end{pmatrix}+c_2\begin{pmatrix}t\\ 2t-1/2\end{pmatrix}+\begin{pmatrix}\frac{2}{t}-\frac{1}{2t^2}-2\ln|t|-2\\ \frac{5}{t}-4\ln|t|-4\end{pmatrix}}$$