1. Given the following differential equations and their corresponding direction field, determine the behavior as $t \to \infty$

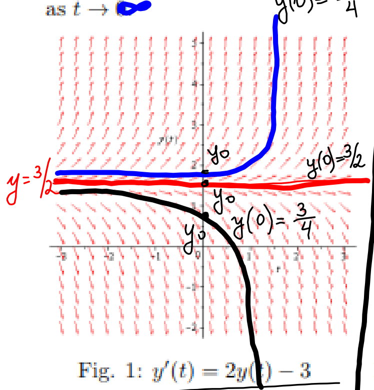


Fig. 1: $y'(t) = 2y(t) - 3$

solutions of initial value problems:

$y' = 2y - 3,\ y(0) = \frac{11}{4} > \frac{3}{2}$

$\lim_{t\to\infty} y(t) = \infty$

$y' = 2y - 3,\ y(0) = \frac{3}{4} < \frac{3}{2}$

$\lim_{t\to\infty} y(t) = -\infty$

$\lim_{t\to-\infty} y(t) = \frac{3}{2}$

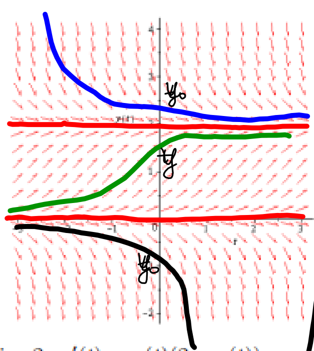


Fig. 2: $y'(t) = y(t)(2 - y(t))$

$y=0,\ y=2$ constant solutions.

solution of initial value problem

$y' = y(2-y),\ y(0) = y_0$

$\lim_{t\to\infty} y(t) = -\infty$, if $y_0 < 0$

$\lim_{t\to\infty} y(t) = 2$, if $0 < y_0 < 2$

$\lim_{t\to\infty} y(t) = 2$, if $y_0 > 2$

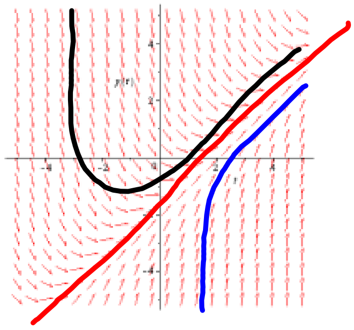


Fig. 3: $y'(t) = t - 1 - y(t)$

$\lim_{t\to\infty} y(t) = \infty$

Can we find the equation of the linear solution of the last differential equation?

Find a linear solution of the equation

$y' = t - 1 - y$

Linear solution is in the form $\boxed{y = at + b}$

where a and b are unknown constants.

$y' = a$

Plug the equations for y and y' back into the original DE.

$\underbrace{a}_{y'} = t - 1 - \underbrace{(at + b)}_{y}$

$a = t - 1 - at - b$

$a = t(1-a) + (-1-b)$

Match the coefficients:

$t: \quad 0 = 1 - a \Rightarrow a = 1$

$1: \quad a = -1 - b \Rightarrow b = -1 - a = -2$

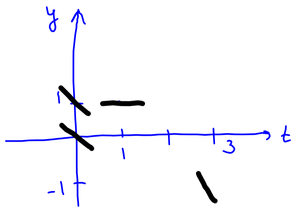
Linear solution of the equation: $\boxed{y(t) = t - 2}$

2. Given the differential equation

$$\frac{dy}{dt} = ty - 1.$$

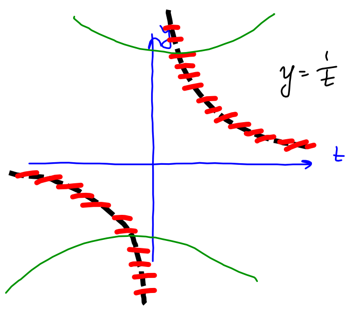
(a) What is the slope of the graph of the solutions at $(0,1)$, at the point $(1,1)$, at the point $(3,-1)$, at the point $(0,0)$?

slope @ $(t,y) = ty-1$
slope @ $(0,1) = -1$
slope @ $(1,1) = 1(1)-1 = 0$
slope @ $(3,-1) = 3(-1)-1 = -4$
slope @ $(0,0) = -1$



(b) Find all the points where the tangents to the solution curves are horizontal?

when the slope $= 0$
$ty-1=0 \Rightarrow y = \frac{1}{t}$



(c) Describe the nature of the critical points.

when integral curves are concave up / concave down?

$y' = ty-1, \quad y = y(t)$
$y'' = (ty)'$
$y'' = y + t\cancel{y'}^{0}$ | working with points when the slope $= y' = 0$.
$y'' = y \Rightarrow$ $y''>0$, if $y>0$ (CU)
$y''<0$, if $y<0$ (CD)

3. The instantaneous rate of change of the temperature T of coffee at time t is proportional to the difference between the temperature M of the air and the temperature T at time t.

(a) Find the mathematical model for the problem.

$T(t)$ is the temperature of coffee @ time t

rate of change is $\frac{dT}{dt}$

$$\frac{dT}{dt} = k(M-T), \quad k \text{ is a constant}$$

(b) Given that the room temperature is 75° and $k = 0.08$, find the solutions to the differential equation.

$M = 75, \ k = 0.08$

$$dt \cdot \frac{\frac{dT}{dt}}{T-75} = \frac{-0.08\,(-75+T)\,dt}{T-75} \quad \Big|\ \text{separable}$$

$$\int \frac{dT}{T-75} = -\int 0.08\,dt$$

$$\ln|T-75| = -0.08t + C, \quad C \text{ is a constant}$$

$$T - 75 = e^{-0.08t+C}$$

$$T = 75 + e^{-0.08t} \cdot e^{C} \to C_1 \text{ - another constant}$$

$$\boxed{T = 75 + C_1 e^{-0.08t}} \quad \text{general solution}$$

(c) The initial temperature of the coffee is 200°F. Find the solution to the problem.

$T(0) = 200$

$T(t) = 75 + C_1 e^{-0.08t}$

$T(0) = 75 + C_1 = 200$

$C_1 = 200 - 75 = 125$

$$\boxed{T(t) = 75 + 125e^{-0.08t}}$$

solution of the initial value problem

$\frac{dT}{dt} = 0.08(75 - T);\ T(0) = 200$

4. Your swimming pool containing 60,000 gal of water has been contaminated by 5 kg of a non toxic dye that leaves a swimmer's skin an unattractive green. The pool's filtering system can take water from the pool, remove the dye, and return the water to the pool at a flow rate of 200 gal/min.

(a) Write down the initial value problem for the filtering process; let $q(t)$ be the amount of dye in the pool at any time t.

$q(0) = 5$ kg

rate of change of $q(t)$ is $\frac{dq}{dt}$

$$\boxed{\frac{dq}{dt} = -200\frac{q(t)}{60,000}, \quad q(0)=5}$$

concentration of the dye

(b) Solve the problem.

$$\frac{dq}{dt} = -\cancel{2}00\frac{q}{3\cancel{60,000}}, \qquad \frac{dt\frac{dq}{dt}}{q} = \frac{-\frac{q}{300}dt}{q}$$

$$\int\frac{dq}{q} = -\int\frac{dt}{300}$$

$$\ln|q| = -\frac{t}{300}+C$$

$$q(t) = e^{-\frac{t}{300}+C}$$

$$q(t) = e^{-\frac{t}{300}}\cdot e^{C} \quad C_1$$

$$\boxed{q(t) = C_1 e^{-\frac{t}{300}}}$$

$q(0) = C_1 = 5$

$$\boxed{q(t) = 5e^{-\frac{t}{300}}} \text{ – mass}$$

(c) You have invited several dozen friends to a pool party that is scheduled to begin in 4 hours. You have also determined that the effect of the dye us imperceptible if its concentration is less than 0.02 g/gal. Is your filtering system capable of reducing the dye concentration to this level within 4 hours? $\boxed{\text{no, it is not}}$

0.02 g/gal $= 2\times10^{-5}$ kg/gal

4 hrs = 240 min.

Concentration of the dye when $t = 240$ min

$$\frac{q(240)}{60,000} = \frac{5e^{-\frac{240}{300}}}{60,000} \approx \frac{2.2466}{60,000} \approx 37444\times10^{-9}$$

$3.7444\times10^{-5} > 2\times10^{-5}$

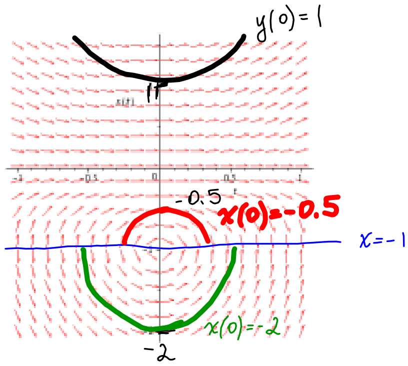
5. The direction field for the differential equation

$$x'(t) = \frac{2tx(t)}{1+x(t)}$$

$1 + x(t) \neq 0$

$x(t) \neq -1$

is given below.



Sketch the graph of the solutions to the initial value problems

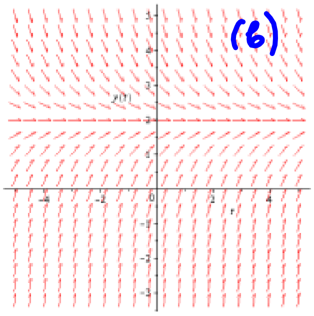
(a) $x(0) = 1$

(b) $x(0) = -2$

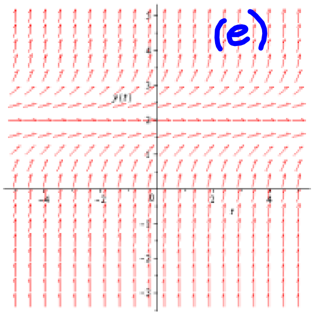
(c) $x(0) = -0.5$

6. Match the direction field to the differential equations

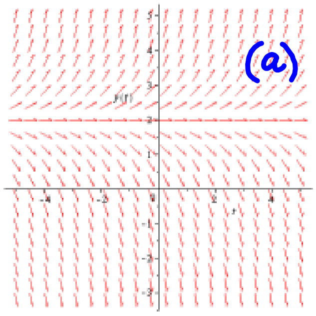
$$\begin{array}{llllll} a) & y' = y - 2 & b) & y' = 2 - y & c) & y' = 2 + y \\ d) & y' = -2 - y & e) & y' = (y-2)^2 & f) & y' = (y+2)^2 \end{array}$$



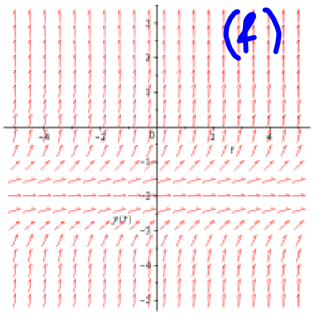
Direction field 1



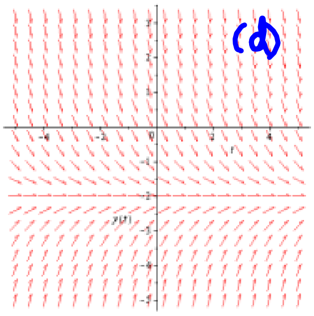
Direction field 2



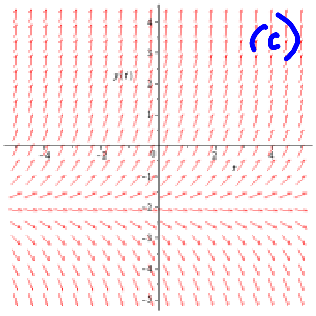
Direction field 3



Direction field 4



Direction field 5



Direction field 6

(a) $y' = y - 2$

slope $= y - 2$

slope $= \begin{cases} >0, & \text{if } y > 2 \\ 0, & \text{if } y = 2 \\ <0, & \text{if } y < 2 \end{cases}$

(b) $y' = 2 - y$

slope $= 2 - y = \begin{cases} >0, & \text{if } y < 2 \\ 0, & \text{if } y = 2 \\ <0, & \text{if } y > 2 \end{cases}$

(c) $y' = 2 + y$

slope $= 2 + y = \begin{cases} >0, & \text{if } y > -2 \\ 0, & \text{if } y = -2 \\ <0, & \text{if } y < -2 \end{cases}$

$2 + y > 0 \Rightarrow y > -2$

(d) $y' = -2 - y$

slope $= -2 - y = \begin{cases} >0, & \text{if } y < -2 \\ 0, & \text{if } y = -2 \\ <0, & \text{if } y > -2 \end{cases}$

7. Given the following differential equations, classify each as an ordinary differential equation, partial differential equation, give the order. If the equation is an ordinary differential equation, say whether the equation is linear or non linear.

(a) $\frac{dy}{dx} = 3y + x^2$. ODE, order 1, linear

(b) $5\frac{d^4y}{dx^4} + y = x(x-1)$. ODE, order 4, linear

(c) $\frac{\partial N}{\partial t} = \frac{\partial^2 N}{\partial r^2} + \frac{1}{r}\frac{\partial N}{\partial r} + kN$. PDE

(d) $\frac{dy}{dx} = 3y + x^2$. ODE, order 1, linear

(e) $\frac{dx}{dt} = x^2 - t$. ODE, order 1, nonlinear

(f) $(1 + y^2)y'' + ty' + y = e^t$. ODE, order 2, nonlinear

(g) $\frac{dy}{dx} + xy^2 = 0$. ODE, order 1, nonlinear

8. Verify that the functions $y_1(t) = t^{-2}$ and $\boxed{y_2(t) = t^{-2}\ln(t)}$ are solutions to the differential equation $t^2y'' + 5ty' + 4y = 0$.

yes, indeed

$$y_2 = t^{-2}\ln t$$
$$y_2' = -2t^{-3}\ln t + t^{-2}\left(\frac{1}{t}\right)$$
$$y_2' = -2t^{-3}\ln t + t^{-3}$$
$$y_2'' = +6t^{-4}\ln t - 2t^{-3}\frac{1}{t} - 3t^{-4}$$
$$= 6t^{-4}\ln t - 2t^{-4} - 3t^{-4}$$
$$y_2'' = 6t^{-4}\ln t - 5t^{-4}$$

$$\cancel{t^2}\underbrace{\left[6t^{\cancel{-4}\,-2}\ln t - 5t^{\cancel{-4}\,-2}\right]}_{y_2''} + 5\cancel{t}\underbrace{\left[-2t^{\cancel{-3}\,-2}\ln t + t^{\cancel{-3}\,-2}\right]}_{y_2'} + 4\underbrace{t^{-2}\ln t}_{y_2} \stackrel{?}{=} 0$$

$$6t^{-2}\ln t - 5t^{-2} - 10t^{-2}\ln t + 5t^{-2} + 4t^{-2}\ln t \stackrel{?}{=} 0$$

$$0 = 0$$

9. (a) Show that $f(x) = (x^2 + Ax + B)e^{-x}$ is a solution to

$$y'' + 2y' + y = 2e^{-x}$$

for all real numbers A and B.

$$y = (x^2 + Ax + B)e^{-x}$$

$$y' = (2x + A)e^{-x} - (x^2 + Ax + B)e^{-x}$$

$$y'' = 2e^{-x} - (2x + A)e^{-x} - (2x + A)e^{-x} + (x^2 + Ax + B)e^{-x}$$

$$y'' = 2e^{-x} - 2(2x + A)e^{-x} + (x^2 + Ax + B)e^{-x}$$

$$\underbrace{2e^{-x} - \cancel{2(2x+A)e^{-x}} + \cancel{(x^2+Ax+B)e^{-x}}}_{y''} + 2\underbrace{\left[\cancel{(2x+A)e^{-x}} - \cancel{(x^2+Ax+B)e^{-x}}\right]}_{y'} + \underbrace{\cancel{(x^2+Ax+B)e^{-x}}}_{y} \stackrel{?}{=} 2e^{-x}$$

$$2e^{-x} = 2e^{-x}$$

$y = (x^2 + Ax + B)e^{-x}$ is a solution for any A and B.

(b) Find a solution that satisfies the initial condition $y(0) = 3$ and $y'(0) = 1$.

$$y(0) = \boxed{B = 3}$$

$$y'(x) = (2x + A)e^{-x} - (x^2 + Ax + B)e^{-x}$$

$$y'(0) = A - B = 1 \Rightarrow \boxed{A = B + 1 = 4}$$

$$\boxed{y(x) = (x^2 + 4x + 3)e^{-x}}$$

10. Determine for which values of r the function t^r is a solution of the differential equation

$$t^2y'' - 4ty' + 4y = 0 \qquad (t>0).$$

$y(t) = t^r$, r is a constant

$y' = rt^{r-1}$

$y'' = r(r-1)t^{r-2}$

Plug into the equation:

$$t^2 \underbrace{r(r-1)t^{r-2}}_{y''} - 4t \underbrace{rt^{r-1}}_{y'} + 4\underbrace{t^r}_{y} = 0$$

$$r(r-1)\underbrace{t^2t^{r-2}}_{t^r} - 4r\underbrace{t\,t^{r-1}}_{t^r} + 4t^r = 0$$

$$\frac{r(r-1)t^r - 4rt^r + 4t^r}{t^r} = \frac{0}{t^r}$$

$$r(r-1) - 4r + 4 = 0$$

$$r^2 - 5r + 4 = 0$$

$$(r-4)(r-1) = 0$$

$$\boxed{r_1 = 4} \ \boxed{r_2 = 1}$$

11. For which values of r is the function $(x-1)e^{-rx}$ solution to $y'' - 6y' + 9y = 0$.

$$y = (x-1)e^{-rx}$$
$$y' = e^{-rx} - r(x-1)e^{-rx}$$
$$y'' = -re^{-rx} - re^{-rx} + r^2(x-1)e^{-rx}$$
$$= -2re^{-rx} + r^2(x-1)e^{-rx}$$

$$\frac{\underbrace{-2re^{-rx} + r^2(x-1)e^{-rx}}_{y''} - 6\left[\underbrace{e^{-rx} - r(x-1)e^{-rx}}_{y'}\right] + 9\underbrace{(x-1)e^{-rx}}_{y}}{e^{-rx}} = \frac{0}{e^{-rx}}$$

$$-2r + r^2(x-1) - 6 + 6r(x-1) + 9(x-1) = 0$$

$$(x-1)\left[r^2 + 6r + 9\right] + (-2r-6) = 0$$

$$r^2 + 6r + 9 = 0 \quad \text{and} \quad -2r - 6 = 0$$
$$(r+3)^2 = 0 \qquad\qquad \boxed{r = -3}$$
$$\boxed{r = -3}$$