1. Find the general solution of the given differential equation.

   (a) $y' + 2ty = 2te^{-t^2}$. linear

   $P(t) = 2t$, $Q(t) = 2te^{-t^2}$

   Integrating factor $\mu(t)$:

   $$\frac{d\mu}{dt} = P(t)\mu$$

   $$\frac{d\mu}{\mu} = \frac{2t\mu\,dt}{\mu} \quad \text{separable}$$

   $$\int \frac{d\mu}{\mu} = \int 2t\,dt$$

   $$\ln|\mu| = t^2 \quad \text{or} \quad \boxed{\mu = e^{t^2}}$$

   Solution $y(t)$:

   $$\mu(t)y(t) = \int Q(t)\mu(t)\,dt$$

   $$e^{t^2}y(t) = \int 2t\underbrace{e^{-t^2}e^{t^2}}_{1}\,dt$$

   $$e^{t^2}y(t) = \int 2t\,dt$$

   $$e^{t^2}y(t) = t^2 + C$$

   $$\boxed{y(t) = e^{-t^2}(t^2 + C)}$$

(b) $2\sqrt{x}\,y' = \sqrt{1-y^2}$, not linear, separable

$y' = \frac{dy}{dx}$

$$\frac{dx\, 2\sqrt{x}\,\frac{dy}{dx}}{\sqrt{1-y^2}\,2\sqrt{x}} = \frac{\sqrt{1-y^2}\,dx}{\sqrt{1-y^2}\,2\sqrt{x}} \quad , 1-y^2 \neq 0 \quad , \quad x \neq 0$$

$y \neq \pm 1$

$x=0$ is a singular point

$$\int \frac{dy}{\sqrt{1-y^2}} = \int \frac{1}{2\sqrt{x}}\,dx$$

$$\arcsin y = \frac{1}{2}\,\frac{x^{1/2}}{1/2} + C$$

$\boxed{\arcsin y = x^{1/2} + C}$ or $\boxed{y = \sin(\sqrt{x} + C)}$

implicit form of the general solution

explicit form of the general solution

Check whether $y=1$ and $y=-1$ are solutions of the equation.

If $y=1$, $y'=0$ plug into the equation

$$2\sqrt{x}\,\cancel{y'}^{\,0} = \sqrt{1-\cancel{y^2}^{\,1}}$$

$$0 = \sqrt{1-1} = 0$$

$\boxed{y=1 \text{ is also a solution of the equation}}$

If $y=-1$, then $y'=0$ and pluging into the equation yelds:

$$2\sqrt{x}\,\cancel{y'}^{\,0} = \sqrt{1-\cancel{y^2}^{\,-1}}$$

$$0 = 0$$

$\boxed{y=-1 \text{ is a solution as well}}$

(c) $y' = 2x \sec y.$ not linear
separable

$$y' = \frac{dy}{dx}$$

$$\frac{dx\,\frac{dy}{dx}}{\sec y} = \frac{2x \sec y}{\sec y}\,dx\,, \quad \sec y \neq 0, \text{ for all } y.$$

$$\int \cos y\,dy = \int 2x\,dx$$

$\sin y = x^2 + C$ general solution, implicit form

$y = \arcsin(x^2 + C)$ general solution, explicit form

(d) $\underline{ty' + y} = \underline{3t\cos t}$, $\quad t > 0$.
$\quad\;\; t \qquad\qquad\; t$

$y' + \frac{y}{t} = 3\cos t$ standard form

$P(t) = \frac{1}{t}$, $Q(t) = 3\cos t$

Integrating factor $\mu(t)$: $\frac{d\mu}{dt} = P(t)\mu(t)$

$\frac{d\mu}{dt} = \frac{\mu}{t}$ separable

$\int \frac{d\mu}{\mu} = \int \frac{dt}{t}$

$\ln|\mu| = \ln|t|$ or $\boxed{\mu = t}$

$\int uv' dt = uv - \int u'v dt$

solution $y(t)$: $\mu(t)\,y(t) = \int Q(t)\mu(t)\,dt$

$t\,y(t) = \int 3t\cos t\,dt$ $\quad \left| \begin{array}{ll} t = u & \cos t = v' \\ u' = 1 & v = \sin t \end{array} \right|$

$= 3t\sin t - 3\int \sin t\,dt$

$t\,y(t) = 3t\sin t + 3\cos t + C$

$$\boxed{y(t) = \frac{3t\sin t + 3\cos t + C}{t}}$$

$$\boxed{y(t) = 3\sin t + \frac{3\cos t}{t} + \frac{C}{t}}$$

2. Find the solution to the initial value problem

(a) $\dfrac{dy}{dx} = 4x^3y - y, \qquad y(1) = -3.$

$$\frac{dx\,\frac{dy}{dx}}{y} = \frac{y(4x^3-1)\,dx}{y}, \quad y \neq 0$$

$$\frac{dy}{y} = (4x^3-1)\,dx$$

$$\ln|y| = x^4 - x + C$$

$$y = e^{x^4-x+C}$$

$$y = e^{x^4-x}\cdot \cancel{e^{c}} \rightarrow C_1$$

$$y = C_1 e^{x^4-x}$$ general solution

plug in $x = 1$

$$y(1) = C_1 e^{1-1} = C_1 e^{0} = \underline{C_1 = -3}$$

$$\boxed{y = -3e^{x^4-x}}$$

(b) $\cancel{d}2\sqrt{x}\dfrac{dy}{dx} = \cos^2 y,dx \quad y(4) = \dfrac{\pi}{4}.$

$$\frac{}{2\sqrt{x}\cos^2 y} \qquad \frac{}{\cos^2 y\, 2\sqrt{x}}$$

$$\int \sec^2 y\, dy = \int \frac{dx}{2\sqrt{x}}$$

$$\tan y = \frac{1}{2}\,\frac{x^{1/2}}{1/2} + C$$

$$\tan y = \sqrt{x} + C \implies y = \arctan(\sqrt{x} + C)$$

$$y(4) = \arctan(\sqrt{4} + C)$$

$$= \arctan(2 + C) = \frac{\pi}{4}$$

$$2 + C = \tan\frac{\pi}{4} = 1 \implies 2 + C = 1$$

$$\boxed{C = -1}$$

$$\boxed{y(t) = \arctan(\sqrt{x} - 1)}$$

(c) $\dfrac{dy}{dt} + \dfrac{2y}{t} = \dfrac{\cos t}{t^2} \qquad y(1) = \dfrac{1}{2}, \qquad t > 0.$

Integrating factor $\mu(t)$:

$$\frac{d\mu}{dt} = P(t)\mu$$

$$\frac{d\mu}{dt} = \frac{2}{t}\mu$$

$$\int \frac{d\mu}{\mu} = \int \frac{2}{t}\,dt$$

$$\ln|\mu| = 2\ln|t| \Rightarrow \mu = t^2$$

solution $y(t)$:

$$t^2 y(t) = \int t^2 \cdot \frac{\cos t}{t^2}\,dt$$

$$t^2 y(t) = \int \cos t\,dt$$

$$t^2 y(t) = \sin t + C$$

$$\boxed{y(t) = \frac{\sin t + C}{t^2} \quad \text{general solution}}$$

$$y(1) = \frac{\sin 1 + C}{1} = \frac{1}{2}$$

$$C = \frac{1}{2} - \sin 1$$

$$\boxed{y(t) = \frac{\sin t + \frac{1}{2} - \sin 1}{t^2}}$$ solution of the initial value problem

3. Consider the initial value problem

$$y' + \overbrace{2}^{P(t)}y = \overbrace{5 - t}^{Q(t)}, \qquad y(0) = y_0$$

Find the value y_0 for which the solution touches, but does not cross the t-axis. at $t = t_1$.

solve the initial value problem first.

Integrating factor $\mu(t)$: $\frac{d\mu}{dt} = 2\mu$

$$\int \frac{d\mu}{\mu} = 2\int dt$$

$$\ln|\mu| = 2t$$

$$\boxed{\mu = e^{2t}}$$

solution $y(t)$: $e^{2t}\, y(t) = \int (5-t)e^{2t}\,dt$ $\quad \left| \begin{array}{ll} u = 5-t & v' = e^{2t} \\ u' = -1 & v = \frac{1}{2}e^{2t} \end{array} \right|$

$$= \frac{1}{2}(5-t)e^{2t} - \int (-1)\frac{1}{2}e^{2t}\,dt$$

$$= \frac{1}{2}(5-t)e^{2t} + \frac{1}{2}\int e^{2t}\,dt$$

$$\frac{e^{2t}\, y(t)}{e^{2t}} = \frac{\frac{1}{2}(5-t)e^{2t} + \frac{1}{4}e^{2t} + C}{e^{2t}}$$

$$\boxed{y(t) = \frac{1}{2}(5-t) + \frac{1}{4} + Ce^{-2t}} \quad \text{general solution}$$

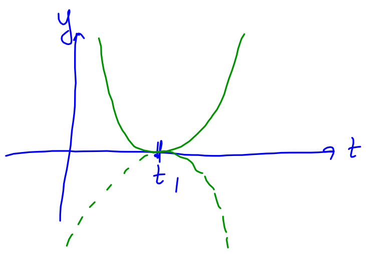
$$y(0) = \frac{5}{2} + \frac{1}{4} + C = y_0$$

$$C = y_0 - \frac{11}{4}$$

$$\boxed{y(t) = \frac{1}{2}(5-t) + \frac{1}{4} + \left(y_0 - \frac{11}{4}\right)e^{-2t}}$$

solution of the initial value problem.

Find t_1 such that $y'(t_1) = 0$ (tangent is horizontal).



$y' + 2y = 5 - t$

plug in $t = t_1$

$\underbrace{y'(t_1)}_{0} + 2\underbrace{y(t_1)}_{0} = 5 - t_1 \Rightarrow 5 - t_1 = 0$

$t_1 = 5.$

initial condition should be $\boxed{y(5) = 0}$

$$y(t) = \frac{1}{2}(5-t) + \frac{1}{4} + Ce^{-2t}$$

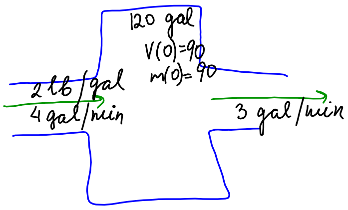
$$y(5) = \frac{1}{4} + Ce^{-10} = 0$$

$$C = -\frac{1}{4}e^{10}$$

$$\boxed{y(t) = \frac{1}{2}(5-t) + \frac{1}{4} - \frac{1}{4}e^{10}e^{-2t}}$$

is going to have horizontal tangent at $t = 5$.

4. A 120-gallon tank initially contains 90 lb of salt dissolved in 90 gallons of water. Brine containing 2 lb/gal of salt flows into the tank at the rate of 4 gal/min, and the well-stirred mixture flows out of the tank at a rate of 3 gal/min. How much salt does the tank contain when it is full?



$m(t)$ is the amount of salt in the tank @ time t.

$$\frac{dm}{dt} = \boxed{\text{rate in}} - \boxed{\text{rate out}}$$

$$\boxed{\text{rate in}} = (\text{concentration})(\text{rate}) = 2 \cdot 4 = 8$$

$$\boxed{\text{rate out}} = (\text{concentration})(\text{rate}) = \frac{m(t)}{90+(4-3)t}\,3 = \frac{3m(t)}{90+t}$$

$$\frac{dm}{dt} = 8 - \frac{3m(t)}{90+t}, \quad m(0) = 90$$

$$\frac{dm}{dt} + \frac{3m(t)}{90+t} = 8 \qquad \text{linear}, \quad P(t) = \frac{3}{90+t}, \quad Q(t) = 8$$

Integrating factor $\mu(t)$:

$$\frac{d\mu}{dt} = \frac{3}{90+t}\mu$$

$$\int \frac{d\mu}{\mu} = \int \frac{3}{90+t}\,dt$$

$$\ln|\mu| = 3\ln|90+t| \Rightarrow \mu = (90+t)^3$$

Solution $m(t)$:

$$(90+t)^3\, m(t) = \int 8(90+t)^3\,dt$$

$$(90+t)^3\, m(t) = 8\,\frac{(90+t)^4}{4} + C$$

$$m(t) = 2(90+t) + \frac{C}{(90+t)^3} \qquad \text{plug into } m(0) = 90$$

$$m(0) = 180 + \frac{C}{27000} = 90$$

$$C = -90(27000)$$

$$C = -81 \times 10^4$$

$$\boxed{m(t) = 2(90+t) - \frac{81 \times 10^4}{(90+t)^3}}$$ solution of the initial value problem.

When the tank is full? $90 + t = 120$

$t = 30$ (min)

$$\boxed{m(30) = 2(90+30) - \frac{81 \times 10^4}{(90+30)^3}}$$

5. In a certain culture of bacteria, the number of bacteria increases sixfold in 10 hrs. How long does it take for the population to double?

$p(t)$ is the population of bacteria @ time t.

$\underbrace{\frac{dp}{dt} = kp(t)}_{\text{separable}}$, k is a constant.

$\int \frac{dp}{p} = \int k\,dt$

$\ln|p| = kt + C$

$p(t) = e^{kt+C} = e^{kt} \cdot \cancel{e^{C}} \to C$

$p(t) = Ce^{kt}$, t is in hours.

$p(10) = 6p(0)$

$p(10) = Ce^{10k} = 6\,p(0) = 6Ce^{0 \cdot k}$

$\cancel{C}e^{10k} = 6\cancel{C}$

$e^{10k} = 6$ or $10k = \ln 6$

$k = \frac{\ln 6}{10}$

$p(t) = Ce^{\frac{\ln 6}{10}t}$

Find t such that $p(t) = 2p(0)$

$\cancel{C}e^{\frac{\ln 6}{10}t} = 2\cancel{C}$

$e^{\frac{\ln 6}{10}t} = 2$

$\frac{\ln 6}{10}t = \ln 2$

$t = \frac{\ln 2}{\frac{\ln 6}{10}} = \frac{10 \ln 2}{\ln 6} \approx \boxed{3.87 \text{ hrs}}$

6. A cake is removed from an over at 210°F and left to cool at room temperature, which is 70°F. After 30 min the temperature of the cake is 140 °F. When will it be 100°F?

$T(t)$ is the temperature of the cake @ time t.

Newton's Law of cooling:

$$\frac{dT}{dt} = k(T-M), \quad M = 70°, \quad T(0)=210, \quad T(30)=140$$

Find t such that $T(t)=100$.

$$\frac{dt\frac{dT}{dt}}{T-70} = \frac{k(T-70)\,dt}{T-70} \text{ separable}$$

$$\int \frac{dT}{T-70} = \int k\,dt \Rightarrow \ln|T-70| = kt + C$$

$$T-70 = e^{kt+C}$$

$$T-70 = e^{kt}\cdot \cancel{e^{c}}\ c$$

$$T-70 = ce^{kt}$$

$$T = 70 + ce^{kt}$$

$$T(0) = 70 + C = 210$$

$$c = 210 - 70 = 140$$

$$T(t) = 70 + 140e^{kt}$$

$$T(30) = 70 + 140e^{30k} = 140$$

$$140e^{30k} = 70$$

$$e^{30k} = \frac{1}{2}$$

$$30k = \ln\frac{1}{2} = -\ln 2$$

$$k = -\frac{\ln 2}{30}$$

$$\boxed{T(t) = 70 + 140e^{-\frac{\ln 2}{30}t}}$$

Find t such that $T(t)=100$

$$70 + 140e^{-\frac{\ln 2}{30}t} = 100$$

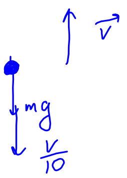
$$140e^{-\frac{\ln 2}{30}t} = 30$$

$$e^{-\frac{\ln 2}{30}t} = \frac{3}{14}$$

$$-\frac{\ln 2}{30}t = \ln\frac{3}{14}$$

$$t = -\frac{\ln\frac{3}{14}}{\frac{\ln 2}{30}} = \boxed{-\frac{30\ln\frac{3}{14}}{\ln 2}} \approx \boxed{66.67 \text{ min}}$$

7. A ball with mass 1kg is thrown upward with initial velocity 20 m/s from the roof of a building 50 m high. A force due to the resistance of the air of $v/10$, where the velocity is measured in m/s, acts on the ball. Find the maximum height above the ground that the ball reaches.



Net force $\vec{F} = m\vec{g} + \frac{\vec{v}}{10}$, $v(0) = 20$

$\vec{F} = m\vec{a}$, $a = \frac{dv}{dt}$

component form: $ma = -mg - \frac{v}{10}$

$m\frac{dv}{dt} = -mg - \frac{v}{10}$, $g = 9.8\ m/sec^2$.

$m = 1$

$$\frac{dv}{dt} = -g - \frac{v}{10}$$

$$\frac{dv}{dt} + \frac{v}{10} = -g \quad \text{linear}, \quad P(t) = \frac{1}{10}, \ Q(t) = -g$$

Integrating factor $\mu(t)$: $\frac{d\mu}{dt} = \frac{\mu}{10}$

$$\int \frac{d\mu}{\mu} = \int \frac{1}{10}\, dt$$

$$\ln|\mu| = \frac{t}{10} \Rightarrow \mu(t) = e^{\frac{t}{10}}$$

Velocity $v(t)$: $v(t)e^{\frac{t}{10}} = -\int g e^{\frac{t}{10}} dt$

$$v(t)e^{\frac{t}{10}} = -10g\, e^{\frac{t}{10}} + C$$

$$v(t) = -10g + Ce^{-\frac{t}{10}}$$

$$v(0) = -10g + C = 20$$

$$C = 20 + 10g$$

$$v(t) = -10g + (20+10g)e^{-\frac{t}{10}}$$

max height is reached when $v(t) = 0$.

$$-10g + (20+10g)e^{-\frac{t}{10}} = 0.$$

$$e^{-\frac{t}{10}} = \frac{10g}{20+10g} = \frac{g}{2+g}$$

$$-\frac{t}{10} = \ln\frac{g}{2+g}$$

$$\boxed{t_{max} = -10 \ln\frac{g}{2+g} \simeq 1.857 \text{ min}}$$

height $h(t) = \int v(t)\,dt$

$$= \int \left[-10g + (20+10g)e^{-\frac{t}{10}}\right] dt$$

$$h(t) = -10gt - 10(20+10g)e^{-\frac{t}{10}} + C, \quad h(0) = 50$$

$$h(0) = -10(20+10g) + C = 50$$

$$-200 - 100g + C = 50$$

$$C = 250 + 100g$$

$$h(t) = -10gt - (200+100g)e^{-\frac{t}{10}} + 250 + 100g$$

max height $\boxed{h_{max} = h(t_{max}) = h(1.857)}$