1. Determine an interval in which the solutions of the following initial value problems are certain to exist.

(a) $y' + \frac{\sin t}{t^2-1}y = \frac{\cot t}{t^2-4t+3}$, $\quad y(2) = -1$.

Thm 2.4.1. For a linear equation $y' + p(t)y = q(t)$.

If both functions $p(t)$ and $q(t)$ are continuous on an interval I, containing t_0, then the initial value problem

$$y' + p(t)y = q(t), \quad y(t_0) = y_0$$

has a unique solution on the interval I.

$p(t) = \frac{\sin t}{t^2-1}$,

$t^2 - 1 \neq 0$

$t \neq \pm 1$

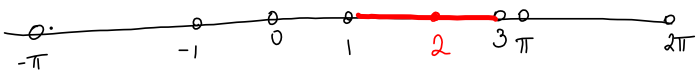
$q(t) = \frac{\cot t}{t^2-4t+3}$

$\cot t$ DNE if $t = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

$t^2 - 4t + 3 \neq 0$

$(t-3)(t-1) \neq 0$

$t \neq 3, t \neq 1$



$\boxed{(1,3)}$

initial condition $y(2) = -1$

(b) $\underline{t(t-4)y' + t^2\ln(t+5)y} = \underline{0,} \qquad y(-3) = 7.$

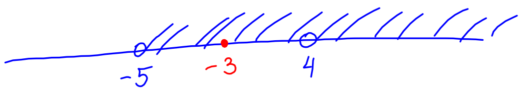
$$\frac{t(t-4)y' + t^2\ln(t+5)y}{t(t-4)} = \frac{0}{t(t-4)}$$

$$y' + \frac{t\ln(t+5)}{t-4}\, y = 0$$

$$p(t) = \frac{t \ln(t+5)}{t-4} \Rightarrow t \neq 4$$

$\ln(t+5) \longrightarrow t+5>0 \longleftarrow$ domain

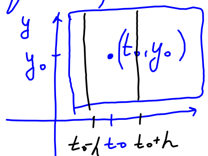
$t > -5$



$\boxed{(-5, 4)}$

$y' = f(t,y),\ y(t_0) = y_0.\ (*)$

Thm 2.4.2. If $f(t,y)$ and $\frac{\partial f}{\partial y}$ are continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point (t_0, y_0), then the solution of the initial value problem $(*)$ exists and is unique for $t_0 - h < t < t_0 + h$, $\gamma < y < \delta$.



2. State where in the ty-plane the hypothesis of theorem 2.4.2 are satisfied.

(a) $y' = \dfrac{\ln(ty)}{1-(t^2+y^2)}$.

$$f(t,y) = \frac{\ln(ty)}{1-(t^2+y^2)}$$

$$\frac{\partial f}{\partial y} = \frac{\frac{1}{\cancel{t}y}\cancel{t}\left[1-t^2-y^2\right] + 2y\ln(ty)}{(1-t^2-y^2)^2}$$

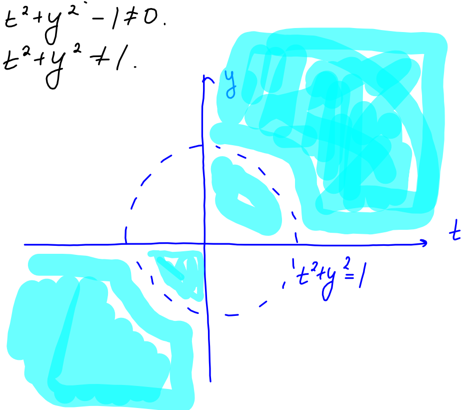
$$= \frac{\frac{1-t^2-y^2}{y} + 2y\ln(ty)}{(1-t^2-y^2)^2}$$

$$= \frac{1-t^2-y^2+2y^2\ln(ty)}{y(1-t^2-y^2)^2}$$

$ty > 0$ either $t > 0,\ y > 0$ or $t < 0,\ y < 0$ $\boxed{\text{1\&3 quadrants}}$

$t^2+y^2-1 \neq 0.$

$t^2+y^2 \neq 1.$

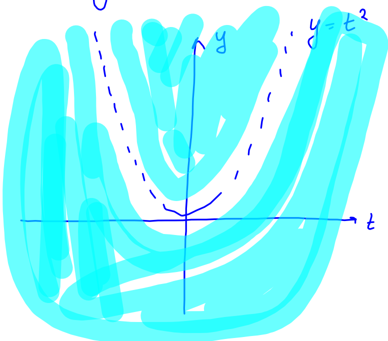


(b) $y' = (t^2 - y)^{1/3}$.

$f(t,y) = (t^2-y)^{1/3}$ continuous on $\mathbb{R}^2$.

$\frac{\partial f}{\partial y} = \frac{1}{3}(t^2-y)^{-2/3}(-1) = -\frac{1}{3}(t^2-y)^{-2/3}$

$t^2 - y \neq 0$, $y \neq t^2$ - eliminate the parabola.



3. Solve the following initial value problems and determine how the interval in which the solution exists depends on the initial value t_0.

(a) $y' = \frac{-4}{t}y, \qquad y(t_0) = y_0$ $\qquad t \neq 0$.

linear

$$\frac{dy}{dt} = -\frac{4}{t}y$$

$$\int \frac{dy}{y} = -\int \frac{4}{t}\,dt$$

$$\ln|y| = -4\ln|t| + \ln C$$

$$\ln|y| = \ln\left(t^{-4}\cdot C\right)$$

$y = \frac{C}{t^4}$ plug into the initial condition:

$y(t_0) = \frac{C}{t_0^4} = y_0$ solve for $C = y_0 t_0^4$

$\boxed{y(t) = \frac{y_0 t_0^4}{t^4}}$ the solution of the initial value problem.

if $t_0 > 0$, then $(0, \infty)$
if $t_0 < 0$, then $(-\infty, 0)$

(b) $y' + y^3 = 0 \qquad y(t_0) = y_0$.

nonlinear. $\quad f(t,y) = -y^3, \quad \frac{\partial f}{\partial y} = -3y^2$ Continuous for all y.

$$\frac{dy}{dt} = -y^3$$

$$\int \frac{dy}{y^3} = -\int dt \quad , y \neq 0.$$

$$\frac{y^{-2}}{-2} = -t + C$$

$$y^{-2} = 2t - 2C$$

$$y = \frac{1}{\sqrt{2t-2C}} \quad \text{plug in } t = t_0$$

$$y(t_0) = \frac{1}{\sqrt{2t_0 - 2C}} = y_0 \quad \text{solve for } C$$

$$\frac{1}{2t_0 - 2C} = y_0^2, \quad 2t_0 - 2C = \frac{1}{y_0^2}, \quad 2C = 2t_0 - \frac{1}{y_0^2}$$

$$C = t_0 - \frac{1}{2y_0^2} = \frac{2t_0 y_0^2 - 1}{2y_0}$$

$$y(t) = \frac{1}{\sqrt{2t - \frac{2t_0 y_0^2 - 1}{y_0}}}, \quad y_0 \neq 0.$$

$$2t - \frac{2t_0 y_0^2 - 1}{y_0} > 0 \quad \text{or} \quad \boxed{t > \frac{2t_0 y_0^2 - 1}{2y_0}, \text{ if } y_0 \neq 0.}$$

If $y_0 = 0$, $y(t_0) = 0$. $\Rightarrow$ $\boxed{y = 0 \text{ only possible solution if } y_0 = 0.}$

4. Verify that both $y_1 = 1 - t$ and $y_2 = \dfrac{-t^2}{4}$ are solutions to the same initial value problem

$$y'(t) = \frac{-t + (t^2 + 4y)^{(1/2)}}{2}, \qquad y(2) = -1.$$

Does it contradict the existence and uniqueness theorem?

$$y'(t) = \frac{-t + \sqrt{t^2+4y}}{2}, \quad y(2) = -1.$$

$y_1 = 1 - t$, $y_1(2) = 1 - 2 = -1$

Plug y_1 back into the equation:

$y_1 = 1 - t$, $y_1' = -1$

$$-1 \stackrel{?}{=} \frac{-t + \sqrt{t^2 + 4(1-t)}}{2}$$

$$-1 \stackrel{?}{=} \frac{-t + \sqrt{t^2 + 4 - 4t}}{2}$$

$$-1 \stackrel{?}{=} \frac{-t + \sqrt{(t-2)^2}}{2}$$

$$-1 \stackrel{?}{=} \frac{-t + t - 2}{2}$$

$$-1 = -1$$

$y_2 = -\frac{t^2}{4}$, $y_2(2) = -\frac{4}{4} = -1$

$y_2' = -\frac{2t}{4} = -\frac{t}{2}.$

Plug y_2, y_2' back into the equation

$$-\frac{t}{2} \stackrel{?}{=} \frac{-t + \sqrt{t^2 + 4\left(-\frac{t^2}{4}\right)}}{2}$$

$$-\frac{t}{2} = -\frac{t}{2}.$$

$f(t,y) = -\dfrac{t + \sqrt{t^2+4y}}{2}$, point $(2,-1)$

$t^2 + 4y \geq 0$ @ $(2,-1)$ $\quad 2^2 + 4(-1) = 0$

$$\frac{\partial f}{\partial y} = -\frac{1}{2} \frac{1}{2\sqrt{t^2+4y}} (4) = -\frac{1}{\sqrt{t^2+4y}}$$

$t^2 + 4y > 0$

plug in $t = 2$ and $y = -1$

$4 + 4(-1) = 0$

$\boxed{\frac{\partial f}{\partial y} \text{ is discontinuous @ } (2,-1).}$

5. Given the differential equation $y' = y(y-2)$

(a) Find the equilibrium solutions

(b) Graph the phase line. Classify each equilibrium solution as either stable, unstable, semistable.

(c) Graph some solutions

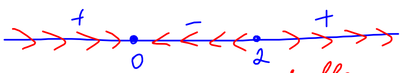
(d) If $y(t)$ is the solution of the equation satisfying the initial condition $y(0) = y_0$, where $-\infty < y_0 < \infty$, find the limit of $y(t)$ when $t \to \infty$

(a) Equilibrium solutions : $y(y-2)=0$

$y=0, \; y=2.$

(b) Phase portrait:

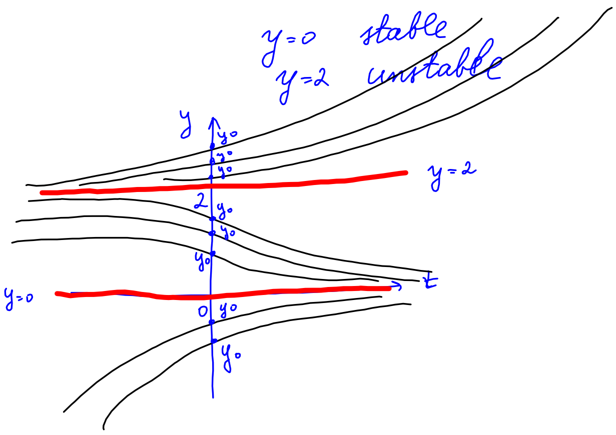
$y(y-2)>0.$



stable unstable

$y=0$ stable

$y=2$ unstable



$y(y-2)=y', \quad y(0)=y_0.$

$y_0<0$	$0<y_0<2$	$y_0>2$
$\lim_{t\to\infty} y(t)=0$	$\lim_{t\to\infty} y(t)=0$	$\lim_{t\to\infty} y(t)=\infty$
$\lim_{t\to-\infty} y(t)=-\infty$	$\lim_{t\to-\infty} y(t)=2$	$\lim_{t\to-\infty} y(t)=2$

6. Given the differential equation

$$y'(t) = y^3 - 2y^2 + y = y(y^2-2y+1) = y(y-1)^2$$

(a) Find the equilibrium solutions

(b) Graph the phase line. Classify each equilibrium solution as either stable, unstable, semistable.

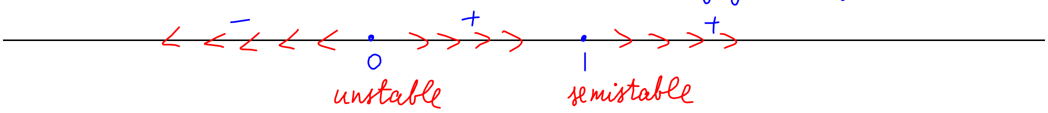
(c) Graph some solutions

(d) If $y(t)$ is the solution of the equation satisfying the initial condition $y(0) = y_0$, where $-\infty < y_0 < \infty$, find the limit of $y(t)$ when $t \to \infty$

(a) Equilibrium solutions: $y(y-1)^2 = 0$

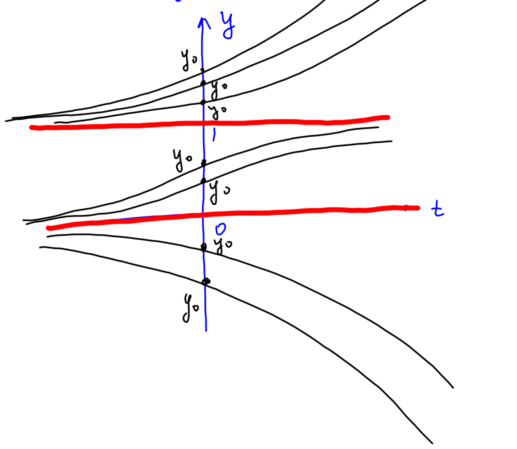
$\boxed{y_1 = 0, \quad y_2 = 1}$

(b) Phase line: $y(y-1)^2 = f(y)$



$y=0$ unstable

$y=1$ semistable



$y' = y(y-1)^2, \; y(0) = y_0$

$y_0 < 0$	$0 < y_0 < 1$	$y_0 > 1$
$\lim_{t\to\infty} y(t) = -\infty$	$\lim_{t\to\infty} y(t) = 1$	$\lim_{t\to\infty} y(t) = \infty$
$\lim_{t\to-\infty} y(t) = 0$	$\lim_{t\to-\infty} y(t)$	$\lim_{t\to-\infty} y(t) = 1$

$M(x,y)+N(x,y)y'=0$ or $M(x,y)dx+N(x,y)dy=0$ is exact if $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

7. Determine if the equations are exact and solve the ones that are:

(a) $(\underbrace{2x+5y}_{M(x,y)})dx+(\underbrace{5x-6y}_{N(x,y)})dy=0.$

$\frac{\partial M}{\partial y}=5, \quad \frac{\partial N}{\partial x}=5$ match, so exact.

Find $F(x,y)$ such that

$$\begin{cases} \int \frac{\partial F}{\partial x}dx \; M(x,y) = \int (2x+5y)dx \Rightarrow F(x,y)=x^2+5xy+g(y) \\ \frac{\partial F}{\partial y}=N(x,y)=5x-6y \qquad \frac{\partial F}{\partial y}=5x+g'(y) \end{cases}$$

should match

$5x-6y=5x+g'(y)$ or $g'(y)=-6y$

$g(y)=-3y^2+C$

update $F(x,y)=x^2+5xy-3y^2+C$

General solution: $\boxed{x^2+5xy-3y^2+C=0}$

(b) $\underbrace{1 + \frac{y}{x}}_{M(x,y)} - \underbrace{\frac{1}{x}}_{N(x,y)} y' = 0.$

$M(x,y) = 1 + \frac{y}{x}$ $\qquad \frac{\partial M}{\partial y} = \frac{1}{x}$

$N(x,y) = -\frac{1}{x}$ $\qquad \frac{\partial N}{\partial x} = \frac{1}{x^2}$

DO NOT MATCH
NOT Exact.

(c) $(\sin(2t)+2y)dy+(2y\cos(2t)-6t^2)dt=0.$

$$\underbrace{(\sin(2t)+2y)}_{N(t,y)}dy+\underbrace{(2y\cos(2t)-6t^2)}_{M(t,y)}dt=0$$

$\frac{\partial M}{\partial y}=2\cos 2t, \quad \frac{\partial N}{\partial t}=2\cos 2t$ exact.

$F(t,y)$ such that

$$\begin{cases} \frac{\partial F}{\partial t}=2y\cos(2t)-6t^2 \\ \int \frac{\partial F}{\partial y}dy = \int(\sin(2t)+2y)dy \end{cases} \Rightarrow F(x,y)=y\sin 2t+y^2+g(t)$$

match!

$$\frac{\partial F}{\partial t}=2y\cos 2t+g'(t)$$

$$2y\cos(2t)-6t^2=2y\cos(2t)+g'(t) \Rightarrow g'(t)=-6t^2,\quad g(t)=-2t^3+C$$

$$F(t,y)=y\sin(2t)+y^2-2t^3+C$$

General solution: $\boxed{y\sin(2t)+y^2-2t^3+C=0}$

8. Show that the equations are not exact. However, if you multiply by the given integrating factor, you can solve the resulting exact equations.

(a) $\underbrace{(x^2+y^2-x)}_{M(x,y)}dx - \underbrace{y}_{N(x,y)}dy = 0 \qquad \mu(x,y) = \frac{1}{x^2+y^2}$

$\mu(x,y)\left[(x^2+y^2-x)dx - y\,dy = 0 \right]$

$\frac{x^2+y^2-x}{x^2+y^2}\,dx - \frac{y}{x^2+y^2}\,dy = 0.$

New $M(x,y) = \frac{x^2+y^2-x}{x^2+y^2} = 1 - \frac{x}{x^2+y^2}$, $\frac{\partial M}{\partial y} = 0 - x\cdot(-1)(x^2+y^2)^{-2}(2y)$

$= \frac{2xy}{(x^2+y^2)^2}$

$N(x,y) = -\frac{y}{x^2+y^2}$

$\frac{\partial N}{\partial x} = -y\cdot(-1)(x^2+y^2)^{-2}(2x) = \frac{2xy}{(x^2+y^2)^2}$ match.

(b) $3(y+1)dx - 2xdy = 0, \qquad \mu(x,y) = \frac{y+1}{x^4}$

$$\frac{y+1}{x^4}\left[3(y+1)\,dx - 2x\,dy\right] = 0.$$

$$\frac{3(y+1)^2}{x^4}\,dx - \frac{2(y+1)}{x^3}\,dy = 0.$$

$$M(x,y) = \frac{3(y+1)^2}{x^4}, \qquad \frac{\partial M}{\partial y} = \frac{6(y+1)}{x^4} \qquad \text{match.}$$

$$N(x,y) = -\frac{2(y+1)}{x^3}, \qquad \frac{\partial N}{\partial x} = -2(y+1)(-3)x^{-4}$$

If $\frac{M_y - N_x}{N}$ depends on x only, then an integrating factor will depend on x only and $\mu(x)$ is a solution of

$$\boxed{\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu}$$

9. Find an integrating factor for the equation

$$\underbrace{(3xy + y^2)}_{M(x,y)} + \underbrace{(x^2 + xy)}_{N(x,y)}y' = 0$$

and then solve the equation.

$$\frac{M_y - N_x}{N} = \frac{(3x+2y)-(2x+y)}{x^2+xy} = \frac{x+y}{x(x+y)} = \frac{1}{x}$$

$\mu = \mu(x)$.

$$\frac{d\mu}{dx} = \frac{1}{x}\mu$$

$$\int \frac{d\mu}{\mu} = \int \frac{dx}{x} \Rightarrow \ln|\mu| = \ln|x| + \cancel{C}^{\,0}$$

$$\mu(x) = x.$$

$$x\left[(3xy+y^2) + (x^2+xy)y'\right] = 0.$$

$$\underbrace{(3x^2y + xy^2)}_{M(x,y)} + \underbrace{(x^3+x^2y)}_{N(x,y)}y' = 0.$$

$$\frac{\partial M}{\partial y} = 3x^2 + 2xy, \quad \frac{\partial N}{\partial x} = 3x^2 + 2xy \qquad \underline{\text{exact}}.$$

$$F(x,y): \begin{cases} \frac{\partial F}{\partial x} = (3x^2y + xy^2) \\ \frac{\partial F}{\partial y} = x^3 + x^2y \end{cases}$$

$$\int (3x^2y + xy^2)\,dx \Rightarrow F(x,y) = x^3y + \frac{x^2}{2}y^2 + g(y)$$

$$\frac{\partial F}{\partial y} = x^3 + x^2y + g'(y)$$

should match

$$x^3 + x^2y = x^3 + x^2y + g'(y) \Rightarrow g'(y) = 0, \quad g(y) = C.$$

$$F(x,y) = \boxed{x^3y + \frac{x^2y^2}{2} + C = 0}$$

general solution.