Math 308 WEEK in REVIEW 8 Spring 2019

1. Find the general solution of the equation $y'' + 6y' + 9y = \dfrac{e^{-3x}}{1+2x}$

1a) Find the general solution of the equation $y'' + 6y' + 9y = \dfrac{e^{-3x}}{1+2x}$

Variation of parameters.

$y''+6y'+9y=0$, $r^2+6r+9=0$, $(r+3)^2=0$, $r=-3$ - repeated root

$y_h(x)=(C_1+C_2x)e^{-3x} = C_1\underbrace{e^{-3x}}_{y_1(x)}+C_2\underbrace{xe^{-3x}}_{y_2(x)}$

$y_1(x)=e^{-3x}$, $y_2(x)=xe^{-3x}$

$W[y_1,y_2]=\begin{vmatrix} e^{-3x} & xe^{-3x} \\ -3e^{-3x} & e^{-3x}-3xe^{-3x}\end{vmatrix} = e^{-3x}(e^{-3x}-\cancel{3xe^{-3x}})+\cancel{3xe^{-3x}\cdot e^{-3x}} = e^{-6x}$

$C_1(x) = -\int \frac{y_2(x)g(x)}{W[y_1,y_2]}dx = -\int \frac{x\cancel{e^{-3x}}}{\cancel{e^{-6x}}}\cdot\frac{\cancel{e^{-3x}}}{1+2x}dx = -\int\frac{x\,dx}{1+2x}$ $\left|\begin{array}{l} u=1+2x \\ x=\frac{u-1}{2} \\ dx=\frac{du}{2}\end{array}\right|$

$= -\int \frac{\frac{u-1}{2}\cdot\frac{du}{2}}{u} = -\frac{1}{4}\int(u-1)u^{-1}du = -\frac{1}{4}\int\left(1-\frac{1}{u}\right)du$

$= -\frac{1}{4}\left(u-\ln|u|\right)+C_3 = -\frac{1}{4}\left(1+2x-\ln|1+2x|\right)+C_3$

$C_2(x) = \int\frac{y_1(x)g(x)}{W[y_1,y_2]}dx = \int\frac{\cancel{e^{-3x}}}{\cancel{e^{-6x}}}\cdot\frac{\cancel{e^{-3x}}}{1+2x}dx = \int\frac{dx}{1+2x} = \ln|1+2x|+C_4$

General solution:

$$\boxed{y(x) = -\frac{1}{4}\left(1+2x-\ln|1+2x|\right)e^{-3x}+C_3e^{-3x}+xe^{-3x}\ln|1+2x|+C_4xe^{-3x}}$$

$mu'' + \gamma u' + ku = F_{external}$

m is the mass (not weight)

k is the spring constant

γ is the damping coefficient.

Hook's Law $F = k\Delta l$

$F_{damping} = \gamma u'$

2. A mass weighing 3 lb stretches a spring 3 in. If the mass is pushed upward, contracting the spring a distance of 1 in. then set in motion with a downward velocity of 2 ft/s, and if there is no damping, find the position u of the mass at any time t. Determine the frequency, period and amplitude of the motion.

$mg = 3$ weight $(g = 32 \text{ ft/sec}^2)$

$m = \frac{3}{g} = \frac{3}{32}$

convert inches into ft

cm into m

Hook's Law $F = k\Delta l$

$3 = k\,\frac{3}{12} \Rightarrow k = 12$

no damping means $\gamma = 0$

$$\boxed{\frac{3}{32}u'' + 12u = 0, \quad u(0) = -\frac{1}{12}, \quad u'(0) = 2}$$

$u'' + 128u = 0$

auxiliary eqn. $r^2 + 128 = 0$ or $r^2 = -128$

$r = \pm i\sqrt{128} \Rightarrow r = \pm i\, 8\sqrt{2}$

$\boxed{u(t) = C_1 \cos 8\sqrt{2}t + C_2 \sin 8\sqrt{2}t}$ – general solution

$u(0) = C_1 = -\frac{1}{12}$

$u'(t) = -8\sqrt{2}C_1 \sin 8\sqrt{2}t + 8\sqrt{2}C_2 \cos 8\sqrt{2}t$

$u'(0) = 8\sqrt{2}C_2 = 2 \Rightarrow C_2 = \frac{2}{8\sqrt{2}} = \frac{1}{4\sqrt{2}}$

solution of the initial value problem

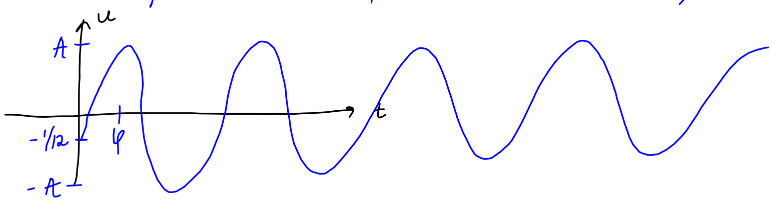
$\boxed{u(t) = -\frac{1}{12}\cos 8\sqrt{2}t + \frac{1}{4\sqrt{2}}\sin 8\sqrt{2}t} = A\cos(8\sqrt{2}t + \varphi)$

Amplitude $A = \sqrt{C_1^2 + C_2^2} = \sqrt{\left(\frac{1}{12}\right)^2 + \left(\frac{1}{4\sqrt{2}}\right)^2} = \frac{\sqrt{22}}{24}$

natural frequency $\omega = 8\sqrt{2}$

period $P = \frac{2\pi}{\omega} = \frac{2\pi}{8\sqrt{2}} = \frac{\pi}{4\sqrt{2}}$

the phase $\varphi = \arctan\frac{C_2}{C_1} = \arctan\left(\frac{\frac{1}{4\sqrt{2}}}{-1/12}\right)$



3. A spring is stretch 10 cm by a force of 3 N. A mass of 2 kg is hung from the spring and is also attached to a viscous damper that exerts a force of 3 N when the velocity of the mass 5 m/s. If the mass is pulled down 5 cm below its equilibrium position and given an initial velocity of 10 cm/s, determine its position u at any time. Find the quasifrequency of the motion.

$$mu'' + \gamma u' + ku = 0, \quad u(0) = 0.05, \ u'(0) = 0.1$$

$$\boxed{m=2}$$

Hook's Law $\quad F = k\,\Delta l \Rightarrow 3 = k(0.1) \Rightarrow \boxed{k=30}$

Damping force $\quad F_d = \gamma u' \Rightarrow 3 = \gamma \cdot 5 \Rightarrow \boxed{\gamma = \frac{3}{5} = 0.6}$

$$2u'' + 0.6u' + 30u = 0$$

$$u'' + 0.3u' + 15u = 0$$

auxiliary eqn: $\quad r^2 + 0.3r + 15 = 0$

roots: $\quad r = \dfrac{-0.3 + \sqrt{(0.3)^2 - 60}}{2} = \dfrac{-0.3 + \sqrt{-59.91}}{2} = -0.15 + i\,\dfrac{\sqrt{59.91}}{2}$

general solution: $\boxed{u(t) = e^{-0.15t}\left(C_1 \cos \frac{\sqrt{59.91}}{2} t + C_2 \sin \frac{\sqrt{59.91}}{2} t\right)}$

$$u(0) = \boxed{C_1 = 0.05}$$

$$u'(t) = -0.15e^{-0.15t}\left(C_1 \cos \frac{\sqrt{59.91}}{2} t + C_2 \sin \frac{\sqrt{59.91}}{2} t\right) + e^{-0.15t}\left(-\frac{\sqrt{59.91}}{2} C_1 \sin \frac{\sqrt{59.91}}{2} t + C_2 \frac{\sqrt{59.91}}{2} \cos \frac{\sqrt{59.91}}{2} t\right)$$

$$u'(0) = -0.15C_1 + C_2 \frac{\sqrt{59.91}}{2} = 0.1$$

$$C_2 = \frac{[0.1 + 0.15(0.05)](2)}{\sqrt{59.91}} = \frac{0.215}{\sqrt{59.91}}$$

solution: $\boxed{u(t) = e^{-0.15t}\left(0.05 \cos \frac{\sqrt{59.91}}{2} t + \frac{0.215}{\sqrt{59.91}} \sin \frac{\sqrt{59.91}}{2} t\right)}$

$$\boxed{\text{quasifrequency} = \frac{\sqrt{59.91}}{2}}$$

4. A mass weighing 4 lb stretches a spring 1.5 in. The mass is given a positive displacement 2 in from its equilibrium position and released with no initial velocity. Assuming that there is no damping and the mass is acted on by an external force of $2\cos 3t$ lb,

$\gamma = 0$

(a) Formulate the initial value problem describing the motion of mass

(b) Solve the initial value problem.

(c) If the given external force is replaced by a force $4\cos\omega t$ of frequency ω, find the value of ω for which resonance occurs.

$mu'' + ku = F_{external}$

$m = \frac{4}{32} = \frac{1}{8}$, Hook's Law $F = k\Delta l$

$4 = k \cdot \frac{1.5}{12} \Rightarrow k = 32$

$$\boxed{\frac{1}{8}u'' + 32u = 2\cos 3t, \quad u(0) = \frac{2}{12}, \quad u'(0) = 0}$$

$u'' + 256u = 16\cos 3t$

$u(t) = u_h(t) + u_p(t)$

homogeneous eqn. $u'' + 256u = 0$

auxiliary eqn. $r^2 + 256 = 0 \Rightarrow r = \sqrt{-256} = \pm 16i$

$$\boxed{u_h(t) = C_1 \cos 16t + C_2 \sin 16t}$$

$u_p(t) = a\cos 3t + b\sin 3t$

$u_p'(t) = -3a\sin 3t + 3b\cos 3t$

$u_p''(t) = -9a\cos 3t - 9b\sin 3t$

$(-9a\cos 3t - 9b\sin 3t) + 256(a\cos 3t + b\sin 3t) = 16\cos 3t$

$247a\cos 3t + 247b\sin 3t = 16\cos 3t$

$\cos 3t$: $247a = 16 \Rightarrow \boxed{a = \frac{16}{247}}$

$\sin 3t$: $247b = 0 \Rightarrow \boxed{b = 0}$

$$\boxed{u_p(t) = \frac{16}{247}\cos 3t}$$

$u = u_h + u_p = C_1\cos 16t + C_2\sin 16t + \frac{16}{247}\cos 3t$ general solution

$u(0) = C_1 + \frac{16}{247} = \frac{1}{6}$

$C_1 = \frac{1}{6} - \frac{16}{247} = \frac{241}{1482}$

$u'(t) = -16C_1\sin 16t + 16C_2\cos 3t - \frac{16}{247}(3)\sin 3t$

$u'(0) = 16C_2 = 0 \Rightarrow \boxed{C_2 = 0}$

$$\boxed{u(t) = \frac{241}{1482}\cos 16t + \frac{16}{247}\cos 3t}$$

(c) $F = 4\cos\omega t$

ω = natural frequency of the system

$$\boxed{\omega = 16}$$

5. A spring is stretched 6 in by a mass that weighs 8 lb. The mass is attached to a dashpot mechanism that has a damping constant of 0.25 lb·s/ft and is acted by an external force of $4\cos 2t$ lb.

   (a) Find the steady-state response of this system.

   (b) if the given mass is replaced by a mass m, determine the value of m for which the amplitude of the steady-state response is maximum.

$mu'' + \gamma u' + ku = F_{external}$

$m = \frac{8}{32} = \frac{1}{4}$, $\quad 8 = k \cdot \frac{6}{12} \Rightarrow k = 16$

$\gamma = 0.25 = \frac{1}{4}$

$\frac{1}{4}u'' + \frac{1}{4}u' + 16u = 4\cos 2t \quad \Big| \quad u(t) = \underbrace{u_h(t)}_{\text{tranrient part}} + \underbrace{u_p(t)}_{\text{steady-state responce.}}$

steady-state responce $= u_p(t)$

$u'' + u' + 64u = 16\cos 2t$

$u_p(t) = a\cos 2t + b\sin 2t$

$u_p'(t) = -2a\sin 2t + 2b\cos 2t$

$u_p''(t) = -4a\cos 2t - 4b\sin 2t$

$(-4a\cos 2t - 4b\sin 2t) + (-2a\sin 2t + 2b\cos 2t) + 64(a\cos 2t + b\sin 2t) = 16\cos 2t$

$(60a + 2b)\cos 2t + (60b - 2a)\sin 2t = 16\cos 2t$

$\cos 2t$: $\quad 60a + 2b = 16 \quad$ or $\quad 30a + b = 8 \Rightarrow 901b = 8 \Rightarrow \boxed{b = \frac{8}{901}}$

$\sin 2t$: $\quad 60b - 2a = 0 \quad$ or $\quad a = 30b$

$\boxed{a = \frac{30 \cdot 8}{901} = \frac{240}{901}}$

steady-state responce $\boxed{u_p(t) = \frac{240}{901}\cos 2t + \frac{8}{901}\sin 2t}$

(b) $\quad mu'' + \frac{1}{4}u' + 16u = 0$

$\underbrace{\frac{\sqrt{\gamma^2 - 4mk}}{2m}}_{\text{quasi frequency when } \gamma=0} = \underbrace{\sqrt{\frac{k}{m}}}_{\text{natural frequency when } \gamma=0}$ solve for m $\Rightarrow \boxed{m \simeq 4}$

Convolution theorem : $H(s)=F(s)\cdot G(s)$, $\mathcal{L}^{-1}\{F(s)\}=f(t)$, $\mathcal{L}^{-1}\{G(s)\}=g(t)$

then $\boxed{\mathcal{L}^{-1}\{H(s)\}=(f*g)(t)=\int_0^t f(t-\tau)g(\tau)\,d\tau=\int_0^t f(\tau)g(t-\tau)\,d\tau}$

6. Find the inverse Laplace transform of the given function by using the convolution theorem.

(a) $F(s)=\dfrac{1}{s^4(s^2+1)}$.

(b) $F(s)=\dfrac{s}{(s+1)^2(s+4)^3}$.

(a) $F(s)=\dfrac{1}{s^4(s^2+1)}=\dfrac{1}{s^4}\cdot\dfrac{1}{s^2+1}$

$\mathcal{L}^{-1}\left\{\dfrac{1}{s^4}\right\}=\dfrac{1}{6}\mathcal{L}^{-1}\left\{\dfrac{3!}{s^4}\right\}=\dfrac{1}{6}t^3$

$\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+1}\right\}=\sin t$

$\mathcal{L}^{-1}\{F(s)\}=\left(\frac{1}{6}t^3\right)*(\sin t)$

$\boxed{=\frac{1}{6}\int_0^t \tau^3\sin(t-\tau)\,d\tau}$

(b) $F(s)=\dfrac{s}{(s+1)^2(s+4)^3}=\dfrac{(s+4)-4}{(s+1)^2(s+4)^3}$

$=\dfrac{\cancel{s+4}}{(s+1)^2(s+4)^{\cancel{3}\,2}}-\dfrac{4}{(s+1)^2(s+4)^3}$

$=\dfrac{1}{(s+1)^2}\cdot\dfrac{1}{(s+4)^2}-4\cdot\dfrac{1}{(s+1)^2}\cdot\dfrac{1}{(s+4)^3}$

$\mathcal{L}^{-1}\left\{\dfrac{1}{(s+1)^2}\right\}=te^{-t}$

$\mathcal{L}^{-1}\left\{\dfrac{1}{(s+4)^2}\right\}=te^{-4t}$

$\mathcal{L}^{-1}\left\{\dfrac{1}{(s+4)^3}\right\}=\dfrac{1}{2}\mathcal{L}^{-1}\left\{\dfrac{2}{(s+4)^3}\right\}=\dfrac{1}{2}t^2e^{-4t}$

$\mathcal{L}^{-1}\{F(s)\}=\left(te^{-t}\right)*\left(te^{-4t}\right)-2\left(te^{-t}\right)*\left(t^2e^{-4t}\right)$

$\boxed{=\int_0^t (t-\tau)e^{-(t-\tau)}\tau e^{-4\tau}\,d\tau-2\int_0^t (t-\tau)e^{-(t-\tau)}\tau^2e^{-4\tau}\,d\tau}$

7. Find the Laplace transform of

(a) $f(t) = \int_0^t (t-\tau)e^{3\tau}\,d\tau = (t)*(e^{3t})$ $\quad \mathcal{L}\{f(t)\} = \mathcal{L}\{t\}\cdot\mathcal{L}\{e^{3t}\} = \boxed{\frac{1}{s^2}\cdot\frac{1}{s-3}}$

(b) $f(t) = \int_0^t e^{\tau}\sin(t-\tau)\,d\tau = (e^{t})*(\sin t)$ $\quad \mathcal{L}\{f(t)\} = \mathcal{L}\{e^{t}\}\cdot\mathcal{L}\{\sin t\} = \boxed{\frac{1}{s-1}\cdot\frac{1}{s^2+1}}$

$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$

8. Find the solution of the initial value problem

(a) $y'' + 3y' + 2y = \begin{cases} \boxed{1}, & 0 \leq t < 10, \\ 0, & 10 \leq t \end{cases}, \qquad y(0) = 0, \quad y'(0) = 0.$

$= \boxed{1} + u_{10}(t)(0-1)$

$= \mathcal{L}\{1 - u_{10}(t)\}$

$\mathcal{L}\{y'' + 3y' + 2y\}$

Let

$\mathcal{L}\{y\} = Y(s)$

$\mathcal{L}\{y'\} = sY(s) - \cancel{y(0)}^{0} = sY(s)$

$\mathcal{L}\{y''\} = s^2Y(s) - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0} = s^2Y(s)$

$\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{1\} - \mathcal{L}\{u_{10}(t)\}$

$s^2Y(s) + 3sY(s) + 2Y(s) = \frac{1}{s} - \frac{e^{-10s}}{s}$

solve for $Y(s)$:

$Y(s)(s^2 + 3s + 2) = \frac{1}{s} - \frac{e^{-10s}}{s}$

$Y(s) = \frac{1}{s(s^2+3s+2)} - \frac{e^{-10s}}{s(s^2+3s+2)}$

$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+3s+2)} - \frac{e^{-10s}}{s(s^2+3s+2)}\right\}$

$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+3s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s+2)(s+1)}\right\} =$

$\frac{1}{s(s+2)(s+1)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1} = \frac{1}{2}\frac{1}{s} + \frac{1}{2}\frac{1}{s+2} - \frac{1}{s+1}$

$1 = A(s+2)(s+1) + Bs(s+1) + Cs(s+2)$

$s=0$: $1 = 2A \Rightarrow \boxed{A = 1/2}$

$s=-1$: $1 = -C \Rightarrow \boxed{C = -1}$

$s=-2$: $1 = 2B \Rightarrow \boxed{B = 1/2}$

$\mathcal{L}^{-1}\left\{\frac{1}{s(s+2)(s+1)}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$

$= \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t}$

$\mathcal{L}^{-1}\left\{\frac{e^{-10s}}{s(s^2+3s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{e^{-10s}}{s(s+1)(s+2)}\right\}$

$\boxed{\mathcal{L}^{-1}\{F(s)e^{-cs}\} = f(t-c)\,u_c(t)}$ $\boxed{\mathcal{L}^{-1}\{F(s)\} = f(t)}$

First, find $\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)(s+2)}\right\} = \frac{1}{2} + \frac{1}{2}e^{-2t} + e^{-t} = f(t)$

$\mathcal{L}^{-1}\left\{\frac{e^{-10s}}{s(s+1)(s+2)}\right\} = f(t-10)\,u_{10}(t)$

$= \left[\frac{1}{2} + \frac{1}{2}e^{-2(t-10)} - e^{-(t-10)}\right]u_{10}(t)$

$$\boxed{y(t) = \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t} - \left[\frac{1}{2} + \frac{1}{2}e^{-2(t-10)} - e^{-(t-10)}\right]u_{10}(t)}$$

(b) $\mathcal{L}\{y'' + 2y' + 2y\} = \mathcal{L}\{\cos t + \delta(t - \pi/2)\}$ $\quad y(0) = 0, \quad y'(0) = 0.$

$\mathcal{L}\{y\} = Y(s)$

$\mathcal{L}\{y'\} = sY(s) - \cancel{y(0)}^{0} = sY(s)$

$\mathcal{L}\{y''\} = s^2Y(s) - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0} = s^2Y(s)$

$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{\cos t\} + \mathcal{L}\{\delta(t-\pi/2)\}$

$s^2Y(s) + 2sY(s) + 2Y(s) = \frac{s}{s^2+1} + e^{-\pi/2 s}$

$Y(s)(s^2+2s+2) = \frac{s}{s^2+1} + e^{-\pi/2 s}$

$Y(s) = \frac{s}{(s^2+1)(s^2+2s+2)} + \frac{e^{-\pi/2 s}}{s^2+2s+2}$

$y(t) = \mathcal{L}^{-1}\{Y(s)\}$

$\frac{s}{(s^2+1)(s^2+2s+2)} = \frac{as+b}{s^2+1} + \frac{cs+d}{s^2+2s+2} \Longrightarrow$ $a = 1/5$, $b = 2/5$, $c = -1/5$, $d = -4/5$

$\frac{s}{(s^2+1)(s^2+2s+2)} = \frac{1}{5}\cdot\frac{s}{s^2+1} + \frac{2}{5}\cdot\frac{1}{s^2+1} - \frac{1}{5}\,\frac{s+4}{s^2+2s+2}$

$= \frac{1}{5}\,\frac{s}{s^2+1} + \frac{2}{5}\,\frac{1}{s^2+1} - \frac{1}{5}\,\frac{(s+1)+3}{(s+1)^2+1}$

$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)(s^2+2s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{5}\,\frac{s}{s^2+1} + \frac{2}{5}\,\frac{1}{s^2+1} - \frac{1}{5}\,\frac{s+1}{(s+1)^2+1} - \frac{3}{5}\,\frac{1}{(s+1)^2+1}\right\}$

$= \frac{1}{5}\cos t + \frac{2}{5}\sin t - \frac{1}{5}e^{-t}\cos t - \frac{3}{5}e^{-t}\sin t$

$\mathcal{L}^{-1}\left\{\frac{e^{-\pi/2 s}}{s^2+2s+2}\right\}$

$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} = e^{-t}\sin t$

$\mathcal{L}^{-1}\left\{\frac{e^{-\pi/2 s}}{s^2+2s+2}\right\} = e^{-(t-\frac{\pi}{2})}\,\cancel{\sin\left(t-\frac{\pi}{2}\right)}^{\cos t}\,u_{\frac{\pi}{2}}(t)$

$$\boxed{y(t) = \frac{1}{5}\cos t + \frac{2}{5}\sin t - \frac{1}{5}e^{-t}\cos t - \frac{3}{5}e^{-t}\sin t + e^{-(t-\frac{\pi}{2})}(\cos t)u_{\pi/2}(t)}$$

(c) $\mathcal{L}\{y'' - y' - 6y\} = \mathcal{L}\{g(t)\}$ $\quad y(0) = 1, \quad y'(0) = 8.$

$\mathcal{L}\{y(t)\} = Y(s)$

$\mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY(s) - 1$

$\mathcal{L}\{y''(t)\} = s^2Y(s) - s\,y(0) - y'(0) = s^2Y(s) - s - 8$

$\mathcal{L}\{g(t)\} = G(s)$

$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 6\mathcal{L}\{y\} = \mathcal{L}\{g(t)\}$

$s^2Y(s) - s - 8 - sY(s) + 1 - 6Y(s) = G(s)$

$Y(s)(s^2 - s - 6) = G(s) + s + 7$

$$Y(s) = \frac{G(s)}{s^2-s-6} + \frac{s+7}{s^2-s-6}$$

$y(t) = \mathcal{L}^{-1}\{Y(s)\}$

$$\mathcal{L}^{-1}\left\{\frac{s+7}{s^2-s-6}\right\} = \mathcal{L}^{-1}\left\{\frac{s+7}{(s-3)(s+2)}\right\}$$

$$\frac{s+7}{(s+3)(s-2)} = \frac{A}{s-3} + \frac{B}{s+2}$$

$s + 7 = A(s+2) + B(s-3)$

$s=-2:\quad 5 = -5B \Rightarrow B = -1$

$s=3:\quad 10 = 5A \Rightarrow A = 2$

$$\mathcal{L}^{-1}\left\{\frac{s+7}{s^2-s-6}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s-3} - \frac{1}{s+2}\right\}$$

$= 2e^{3t} - e^{-2t}$

$$\mathcal{L}^{-1}\left\{\frac{G(s)}{s^2-s-6}\right\} = \mathcal{L}^{-1}\left\{G(s)\cdot\frac{1}{s^2-s-6}\right\}$$

$\mathcal{L}^{-1}\{G(s)\} = g(t)$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-s-6}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-3)(s+2)}\right\}$$

$$\frac{1}{(s-3)(s+2)} = \left(\frac{1}{s-3} - \frac{1}{s+2}\right)\frac{1}{5}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-s-6}\right\} = \frac{1}{5}\left[\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}\right]$$

$= \frac{1}{5}\left(e^{3t} - e^{-2t}\right)$

$$\mathcal{L}^{-1}\left\{\frac{G(s)}{s^2-s-6}\right\} = g(t) * \left(\frac{1}{5}(e^{3t} - e^{-2t})\right)$$

$$= \frac{1}{5}\int_0^t \left(e^{3\tau} - e^{-2\tau}\right) g(t-\tau)\,d\tau$$

$$\boxed{y(t) = 2e^{3t} - e^{-2t} + \frac{1}{5}\int_0^t \left(e^{3\tau} - e^{-2\tau}\right) g(t-\tau)\,d\tau}$$

#~~2~~9. $y''+xy'+2y=0$.

(a) $y=\sum_{n=0}^{\infty} a_n x^n$, $y'=\sum_{n=1}^{\infty} a_n n x^{n-1}$, $y''=\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}$

Plug into the equation:

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}+x\sum_{n=1}^{\infty} a_n n x^{n-1}+2\sum_{n=0}^{\infty} a_n x^n=0$$

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n+\sum_{n=1}^{\infty} a_n n x^n+2\sum_{n=0}^{\infty} a_n x^n=0.$$

$$a_{0+2}(0+2)(0+1)x^0+\sum_{n=1}^{\infty} a_{n+2}(n+2)(n+1)x^n+\sum_{n=1}^{\infty} a_n n x^n$$
$$+2a_0x^0+\sum_{n=0}^{\infty} 2a_n x^n=0.$$

$$2a_2+2a_0+\sum_{n=1}^{\infty}\left[a_{n+2}(n+2)(n+1)+na_n+2a_n\right]x^n=0.$$

$2a_2+2a_0=0 \Rightarrow a_2=-a_0$

$a_{n+2}(n+2)(n+1)+(n+2)a_n=0 \Rightarrow \boxed{a_{n+2}=-\frac{a_n}{n+1}}$ $n\geq 0$

recurrence relation.

a_0

a_1

$a_2=-a_0$

$a_3=-\frac{a_1}{2}$ ~~4~~

$a_4=-\frac{a_2}{3}=\frac{a_0}{3}$

$a_5=-\frac{a_3}{4}=\frac{a_1}{2\cdot 4}$

$$a_{2n+1}=(-1)^n\frac{a_1}{2^n\cdot n!}$$

$$a_{2n}=(-1)^n\frac{a_0}{1\cdot 3\cdot\ldots\cdot(2n-1)}$$

$$y(t)=\sum_{n=0}^{\infty}(-1)^n\frac{a_1}{2^n n!}x^{2n+1}$$
$$+\sum_{n=0}^{\infty}(-1)^n\frac{a_0}{1\cdot 3\cdot\ldots\cdot(2n-1)}x^{2n}$$

$$=a_1\underbrace{\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2^n\cdot n!}}_{y_1(x)}+a_0\underbrace{\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{1\cdot 3\cdot\ldots\cdot(2n-1)}}_{y_2(x)}$$

(b) $y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^n n!} = x - \frac{x^3}{2} + \frac{x^5}{8} - \frac{x^7}{48} + \ldots$

$y_1(0) = 0, \quad y_1'(x) = 1 - \frac{3x^2}{2} + \frac{5x^4}{8} - \frac{7x^6}{48} + \ldots$

$y_1'(0) = 1$

$y_2(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{1 \cdot 3 \cdot \ldots \cdot (2n-1)} = 1 - \frac{x^2}{1} + \frac{x^4}{3} - \frac{x^6}{15} + \ldots$

$y_2(0) = 1, \quad y_2'(x) = -2x + \frac{4x^3}{3} - \frac{6x^5}{15} + \ldots$

$y_2'(0) = 0.$

$W[y_1, y_2](0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0.$