1. Transform the given equation into a system of first order equation, then in matrix notation.

   (a) $e^t y'' + t^2 y' - (\sin t) y = 3 \arctan t, \quad y(0) = 5, \quad y'(0) = 3.$

   (b) $y'' - \cos t y' + 3ty = 0.$

(a) $e^t y'' + t^2 y' - (\sin t) y = 3\arctan t, \quad y(0)=5, \; y'(0)=3.$

Introducing new functions:

$$\begin{cases} x_1(t) = y(t) \Rightarrow x_1'(t) = y'(t) = x_2(t) \\ x_2(t) = y'(t) \Rightarrow x_2'(t) = y''(t) \end{cases}$$

$$\boxed{\begin{cases} x_1'(t) = x_2(t) \\ x_2'(t) = 3e^{-t}\arctan t - t^2 e^{-t} x_2(t) + e^{-t}(\sin t) x_1(t) \end{cases}}$$

$y'' = \left(3\arctan t - t^2 y' + (\sin t) y\right) e^{-t}$

or

$y'' = 3e^{-t}\arctan t - t^2 e^{-t} \cancel{y'}^{\,x_2(t)} + e^{-t}(\sin t)\cancel{y}^{\,x_1(t)}$

Initial conditions:

$y(0) = 5 \Rightarrow y(0) = \underline{x_1(0) = 5},$

$y'(0) = 3 \Rightarrow y'(0) = \underline{x_2(0) = 3},$

$$\boxed{\begin{aligned} x_1(0) &= 5 \\ x_2(0) &= 3 \end{aligned}}$$

(b) $y'' - (\cos t)y' + 3ty = 0.$

Clone

$y'' = (\cos t)\, y' - 3ty$

New functions
$$\begin{cases} x_1(t) = y(t) & \Rightarrow \quad x_1'(t) = y'(t) = x_2(t) \\ x_2(t) = y'(t) & \Rightarrow \quad x_2'(t) = y''(t) = (\cos t)\, y' - 3ty \\ & \qquad\qquad\qquad\quad = (\cos t)\, x_2(t) - 3t\, x_1(t) \end{cases}$$

$$\boxed{\begin{cases} x_1'(t) = x_2(t) \\ x_2'(t) = (\cos t)\, x_2(t) - 3t\, x_1(t) \end{cases}}$$

2. Transform the given system into a single equation of second order. Find x_1 and x_2 that satisfies the initial conditions when initial conditions are given:

(a) $\begin{cases} x_1' = x_1 - 2x_2, & x_1(0) = -1 \\ x_2' = 3x_1 - 4x_2, & x_2(0) = 2 \end{cases}$.

(b) $\begin{cases} x_1' + 2x_2' = 4x_1 + 5x_2 \\ 2x_1' - x_2' = 3x_1 \end{cases}$

(a) $\begin{cases} x_1' = x_1 - 2x_2 \\ x_2' = 3x_1 - 4x_2 \end{cases} \Rightarrow (x_1)' = \left(\dfrac{x_2' + 4x_2}{3}\right)'$

$$x_1' = \frac{x_2'' + 4x_2'}{3}$$

Plug into the 1st equation:

$$3\Bigg(\underbrace{\frac{x_2'' + 4x_2'}{3}}_{x_1'}\Bigg) = \Bigg(\underbrace{\frac{x_2' + 4x_2}{3}}_{x_1} - 2x_2\Bigg)3$$

$$x_2'' + 4x_2' = x_2' + 4x_2 - 6x_2$$

$$x_2'' + 3x_2' + 2x_2 = 0, \quad x_2(0) = 2.$$

Need $x_2'(0) = 3x_1(0) - 4x_2(0)$ (follows from the 2nd equation).

$$= 3(-1) - 4(2)$$

$$= -3 - 8$$

$$x_2'(0) = -11$$

Initial value problem: $\boxed{x_2'' + 3x_2' + 2x_2 = 0, \quad x_2(0) = 2, \; x_2'(0) = -11}$

$$r^2 + 3r + 2 = 0$$

$$(r_1+2)(r+1) = 0 \Rightarrow r_1 = -2, \; r_2 = -1$$

$$x_2(t) = C_1 e^{-t} + C_2 e^{-2t}, \quad x_2'(t) = -C_1 e^{-t} - 2C_2 e^{-2t}$$

Plug into the initial conditions:

$$\begin{aligned} x_2(0) &= C_1 + C_2 = 2 \\ x_2'(0) &= -C_1 - 2C_2 = -11 \end{aligned} \Rightarrow -C_2 = -9 \Rightarrow \boxed{C_2 = 9} \quad \boxed{C_1 = -7}$$

$$\boxed{x_2(t) = -7e^{-t} + 9e^{-2t}}$$

Now find $x_1(t)$:

$$x_1(t) = \frac{x_2'(t) + 4x_2(t)}{3} = \frac{\overbrace{7e^{-t} - 18e^{-2t}}^{x_2'} + 4(-7e^{-t} + 9e^{-2t})}{3}$$

$$x_1(t) = \frac{-21e^{-t} + 18e^{-2t}}{3}$$

$$\boxed{x_1(t) = -7e^{-t} + 6e^{-2t}}$$

(b) $\begin{cases} x_1' + 2x_2' = 4x_1 + 5x_2 \\ 2(2x_1' - x_2' = 3x_1) \end{cases}$

$x_2' = 2x_1' - 3x_1$

$\begin{cases} x_1' + 2x_2' = 4x_1 + 5x_2 \\ + \; 4x_1' - 2x_2' = 6x_1 \end{cases}$

$\frac{5x_1'}{5} = \frac{10x_1 + 5x_2}{5} \Rightarrow x_1' = 2x_1 + x_2$

$(x_2)' = (x_1' - 2x_1)'$

$x_2' = x_1'' - 2x_1 \Rightarrow x_1'' - 2x_1 = 2x_1' - 3x_1$

$\boxed{x_1'' - 2x_1' + x_1 = 0.}$

3. Verify that $\psi = \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix}$ is solution to

$$\psi' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \psi$$

$$\psi'(t) = \begin{pmatrix} -3e^{-3t} & 2e^{2t} \\ 12e^{-3t} & 2e^{2t} \end{pmatrix}$$

Plug into the system:

$$\begin{pmatrix} -3e^{-3t} & 2e^{2t} \\ 12e^{-3t} & 2e^{2t} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-3t} - 4e^{-3t} & e^{2t} + e^{2t} \\ 4e^{-3t} + 8e^{-3t} & 4e^{2t} - 2e^{2t} \end{pmatrix}$$

$$\begin{pmatrix} -3e^{-3t} & 2e^{2t} \\ 12e^{-3t} & 2e^{2t} \end{pmatrix} = \begin{pmatrix} -3e^{-3t} & 2e^{2t} \\ 12e^{-3t} & 2e^{2t} \end{pmatrix}$$

$\begin{pmatrix} 4 & -2 \end{pmatrix}$

4. Given the matrices

$$A = \begin{pmatrix} -2 & 1 & 2 \\ 1 & 0 & -3 \\ 2 & -1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -1 & -1 \\ -2 & 1 & 0 \end{pmatrix}$$

(a) Find $3A - 2B$

(b) Calculate $\det(A)$.

(c) Find $A^2 - AB$.

(a) $3A - 2B = 3\begin{pmatrix} -2 & 1 & 2 \\ 1 & 0 & -3 \\ 2 & -1 & 1 \end{pmatrix} - 2\begin{pmatrix} 1 & 2 & 3 \\ 3 & -1 & -1 \\ -2 & 1 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 3(-2)-2(1) & 3(1)-2(2) & 3(2)-2(3) \\ 3(1)-2(3) & 3(0)-2(-1) & 3(-3)-2(-1) \\ 3(2)-2(-2) & 3(-1)-2(1) & 3(1)-2(0) \end{pmatrix} = \begin{pmatrix} -8 & -1 & 0 \\ -3 & 2 & -7 \\ 10 & 5 & 3 \end{pmatrix}$$

(b) $\det A = \left| \begin{array}{ccc|cc} -2 & 1 & 2 & -2 & 1 \\ 1 & 0 & -3 & 1 & 0 \\ 2 & -1 & 1 & 2 & -1 \end{array} \right| = -2(0)(1) + 1(-3)(2) + 2(1)(-1) - 2(0)(2) - (-1)(-2)(-3) - (1)(1)(1)$

$= -6 - 2 + 6 - 1 = \boxed{-3}$

(c) $A^2 - AB = A(A-B)$

$$= \begin{pmatrix} -2 & 1 & 2 \\ 1 & 0 & -3 \\ 2 & -1 & 1 \end{pmatrix} \left[\begin{pmatrix} -2 & 1 & 2 \\ 1 & 0 & -3 \\ 2 & -1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 3 \\ 3 & -1 & -1 \\ -2 & 1 & 0 \end{pmatrix} \right]$$

$$= \begin{pmatrix} -2 & 1 & 2 \\ 1 & 0 & -3 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} -3 & -1 & -1 \\ -2 & 1 & -2 \\ 4 & -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2(-3)+1(-2)+2(4) & (-2)(-1)+1(1)+2(-2) & -2(-1)+1(-2)+2(1) \\ 1(-3)+0(-2)-3(4) & 1(-1)+1(0)-3(-2) & 1(-1)+(0)(-2)-3(1) \\ 2(-3)-1(-2)+1(4) & 2(-1)-1(1)+1(-2) & 2(-1)-1(-2)+1(1) \end{pmatrix}$$

$$= \begin{pmatrix} 12 & -1 & 2 \\ -15 & 5 & -4 \\ 0 & -5 & 1 \end{pmatrix}$$

5. Given the matrices and vectors

$$A = \begin{pmatrix} -2 & 0 & 1 \\ 1 & -1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 \\ -1 & 0 \\ 3 & -1 \end{pmatrix}, \quad X(t) = \begin{pmatrix} 2t \\ e^{-t} \end{pmatrix}, \quad Y(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}.$$

(a) Find AB and BA

(b) Find AX, BX, AY, BY if possible.

$$AB = \begin{pmatrix} -2 & 0 & 1 \\ 1 & -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ -1 & 0 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} -2(1)+0(-1)+1(3) & -2(3)+0(0)+1(-1) \\ 1(1)-1(-1)+3(3) & 1(3)-1(0)+3(-1) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -7 \\ 11 & 0 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & 3 \\ -1 & 0 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} -2 & 0 & 1 \\ 1 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 1(-2)+3(1) & 1(0)+3(-1) & 1(1)+3(3) \\ -1(-2)+0(1) & -1(0)+0(-1) & (-1)(1)+0(3) \\ 3(-2)+(-1)1 & 3(0)+(-1)(-1) & 3(1)-1(3) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -3 & 10 \\ 2 & 0 & -1 \\ -7 & 1 & 0 \end{pmatrix}$$

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in general, $AB \neq BA$.

$$A = \begin{pmatrix} -2 & 0 & 1 \\ 1 & -1 & 3 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} 2t \\ e^{-t} \end{pmatrix}, \quad \vec{Y}(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}$$

$$A\vec{X} = \begin{pmatrix} -2 & 0 & 1 \\ 1 & -1 & 3 \end{pmatrix}\begin{pmatrix} 2t \\ e^{t} \end{pmatrix} \quad \text{not possible.}$$

$$A\vec{Y} = \begin{pmatrix} -2 & 0 & 1 \\ 1 & -1 & 3 \end{pmatrix}\begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix} = \begin{pmatrix} -2\cos t + 0(\sin t) + t \\ \cos t - \sin t + 3t \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 3 \\ -1 & 0 \\ 3 & -1 \end{pmatrix}$$

$$B\vec{X} = \begin{pmatrix} 1 & 3 \\ -1 & 0 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 2t \\ e^{-t} \end{pmatrix} = \begin{pmatrix} 2t + 3e^{-t} \\ -2t + 0\cdot e^{-t} \\ 3(2t) - 1e^{-t} \end{pmatrix} = \begin{pmatrix} 2t + 3e^{-t} \\ -2t \\ 6t - e^{-t} \end{pmatrix}$$

$$B\vec{Y} = \begin{pmatrix} 1 & 3 \\ -1 & 0 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix} \quad \text{not possible}$$

Vectors $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ are linearly independent if $c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_n\vec{x}_n = \vec{0}$ if and only if $c_1 = c_2 = \dots = c_n$

If $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ are not linearly independent, then they are linearly dependent.

If $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ are linearly independent, then $W[\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n] \neq 0$

$$\vec{x}_1 = \begin{pmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1n} \end{pmatrix}, \quad \vec{x}_2 = \begin{pmatrix} x_{21} \\ x_{22} \\ \vdots \\ x_{2n} \end{pmatrix}, \dots, \quad \vec{x}_n = \begin{pmatrix} x_{n1} \\ x_{n2} \\ \vdots \\ x_{nn} \end{pmatrix}$$

$$W[\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n] = \begin{vmatrix} x_{11} & x_{21} & \dots & x_{n1} \\ x_{12} & x_{22} & \dots & x_{n2} \\ \vdots & \vdots & & \vdots \\ x_{1n} & x_{2n} & \dots & x_{nn} \end{vmatrix}$$

6. Determine whether the vectors are linearly independent. If they are linearly dependent, find a linear relation among them:

(a) $X_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}.$

(b) $X_1 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}.$

linearly independent

find c such that $\vec{x}_1 = c\vec{x}_2$, $c \neq 0$.

$$\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = c\begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 2 = 5c \\ 3 = c \\ 1 = 0 \end{pmatrix} \text{ not possible.}$$

(a) $$W[\vec{x}_1, \vec{x}_2, \vec{x}_3] = \begin{vmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$\vec{x}_1, \vec{x}_2, \vec{x}_3$ are linearly dependent

means, that there are constants a and b such that

$$\vec{x}_1 = a\vec{x}_2 + b\vec{x}_3$$

$$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = a\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -b \\ a+2b \\ 0 \end{pmatrix} \implies \begin{cases} -b = 2 & \implies b = -2 \\ a + 2b = 1 & \implies a = 1 - 2b = 1 - 2(-2) = 5 \end{cases}$$

$$\boxed{\vec{x}_1 = 5\vec{x}_2 - 2\vec{x}_3}$$

7. Are the vector functions linearly independent? If they are linearly dependent, find a linear relation among them.

(a) $X_1(t) = \begin{pmatrix} e^{-3t} \\ -4e^{-3t} \end{pmatrix}$ $X_2(t) = \begin{pmatrix} e^{-3t} \\ e^{-3t} \end{pmatrix}$. (linearly independent)

(b) $X_1(t) = \begin{pmatrix} 2e^{t} \\ -e^{t} \end{pmatrix}$, $X_2(t) = \begin{pmatrix} e^{t} \\ 2e^{t} \end{pmatrix}$ $X_3(t) = \begin{pmatrix} 3te^{t} \\ te^{t} \end{pmatrix}$.

(a) $$W[\vec{x}_1, \vec{x}_2] = \begin{vmatrix} e^{-3t} & e^{-3t} \\ -4e^{-3t} & e^{-3t} \end{vmatrix} = e^{-3t}(e^{-3t}) + 4e^{-3t}(e^{-3t})$$
$$= 5e^{-3t} \neq 0.$$

(b) find constants a and b such that

$\vec{x}_1 = a\vec{x}_2 + b\vec{x}_3$

$\vec{x}_1 = \begin{pmatrix} 2e^{t} \\ -e^{t} \end{pmatrix}$, $\vec{x}_2 = \begin{pmatrix} e^{t} \\ 2e^{t} \end{pmatrix}$, $\vec{x}_3 = \begin{pmatrix} 3t\ e^{t} \\ te^{t} \end{pmatrix}$

linearly independent.

simple observation,

$(\vec{x}_1 + \vec{x}_2)\, t = \vec{x}_3$

A number λ is an eigenvalue for a matrix A, if there is a nonzero vector $\vec{v}$ such that

$$\boxed{A\vec{v} = \lambda\vec{v}}$$

$\vec{v}$ is a eigenvector corresponding ~~to λ.~~

To find eigen values, we solve the equation

$$\boxed{\det(A-\lambda I) = 0}$$

8. Find the eigenvalues and eigenvectors of the given matrix

(a) $A = \begin{pmatrix} -2 & 1 \\ -1 & -2 \end{pmatrix}$

$$\det(A-\lambda I) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A$$

$$\operatorname{tr} A = -2-2 = -4$$

$$\det A = \begin{vmatrix} -2 & 1 \\ -1 & -2 \end{vmatrix} = 4+1 = 5$$

$$\lambda^2 + 4\lambda + 5 = 0$$

$$\boxed{\lambda_1 = \frac{-4+\sqrt{16-4(5)}}{2} = -2+i}, \quad \boxed{\lambda_2 = -2-i} \quad \text{eigenvalues}$$

Corresponding eigenvectors:

$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ it is a solution of the system

$$(A-\lambda I)\vec{v} = \vec{0}$$

$$\left[\begin{pmatrix} -2 & 1 \\ -1 & -2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\vec{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2-(-2+i) & 1 \\ -1 & -2-(-2+i) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{solve for } v_1 \text{ and } v_2.$$

$$\begin{cases} -i\, v_1 + v_2 = 0 \\ -v_1 - i\, v_2 = 0 \end{cases} \qquad \begin{cases} +i(-v_1 - i v_2) = 0 \\ -v_1 - i v_2 = 0 \end{cases}$$

$$v_1 = -i v_2$$

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -i\, v_2 \\ v_2 \end{pmatrix} \overset{v_2=1}{=} \boxed{\begin{pmatrix} -i \\ 1 \end{pmatrix} \text{ corresponds to } -2+i}$$

$$\boxed{\begin{pmatrix} i \\ 1 \end{pmatrix} \text{ corresponds to } \lambda_2 = -2-i}$$

(b) $A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$.

eigenvalues! $\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix}$

$= (3-\lambda)(-\lambda)(3-\lambda) + 16 + 16 + 16\lambda - 4(3-\lambda) - 4(3-\lambda)$

$= -\lambda^3 + 6\lambda^2 + 15\lambda + 8 = 0$

$\lambda^3 - 6\lambda^2 - 15\lambda - 8 = 0$

$\lambda = 1$: $\quad 1-6-15-8 \neq 0$

eigenvalue $\lambda = -1$: $\quad -1-6+15-8 = 0$

$\lambda^3 - 6\lambda^2 - 15\lambda - 8 = (\lambda+1)(\lambda^2 - 7\lambda - 8)$

$= (\lambda+1)(\lambda-8)(\lambda+1)$

$= (\lambda+1)^2(\lambda-8) = 0$

$\lambda_1 = 8$

$\lambda_2 = -1$ - repeated root.

$$\begin{array}{r|l} & \lambda^2 - 7\lambda - 8 \\ \hline \lambda+1 & \lambda^3 - 6\lambda^2 - 15\lambda - 8 \\ & \underline{\lambda^3 + \lambda^2} \\ & -7\lambda^2 - 15\lambda \\ & \underline{-7\lambda^2 - 7\lambda} \\ & -8\lambda - 8 \end{array}$$

Eigenvectors. $\lambda_1 = 8$ $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$\vec{v}$ is a solution of $(A - \lambda_1 I)\vec{v} = \vec{0}$

$$\begin{pmatrix} 3-8 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & 3-8 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

solve for v_1, v_2, v_3

$$\begin{cases} -5v_1 + 2v_2 + 4v_3 = 0 \\ 2v_1 - 8v_2 + 2v_3 = 0 \\ 4v_1 + 2v_2 - 5v_3 = 0 \end{cases} \Rightarrow \begin{cases} -5v_1 + 2v_2 + 4v_3 = 0 \\ v_1 - 4v_2 + v_3 = 0 \\ 4v_1 + 2v_2 - 5v_3 = 0 \end{cases}$$

1st + 3rd $\begin{cases} v_1 - 4v_2 + v_3 = 0 \\ -v_1 + 4v_2 - v_3 = 0 \\ 4v_1 + 2v_2 - 5v_3 = 0 \end{cases}$

$$\begin{cases} v_1 - 4v_2 + v_3 = 0 \\ 4v_1 + 2v_2 - 5v_3 = 0 \end{cases}$$

$$v_1 = 4v_2 - v_3$$

$$4(4v_2 - v_3) + 2v_2 - 5v_3 = 0$$

$$16v_2 - 4v_3 + 2v_2 - 5v_3 = 0$$

$$18v_2 - 9v_3 = 0$$

$$\boxed{v_3 = 2v_2}$$

$$v_1 = 4v_2 - 2v_2 = 2v_2$$

$$\vec{v} = \begin{pmatrix} 2v_2 \\ v_2 \\ 2v_2 \end{pmatrix} = v_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \overset{v_2 = 1}{=} \boxed{\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \text{ corresponds to } \lambda_1 = 8}$$

$\lambda_2 = -1$ eigenvector $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ is a solution of $(A - \lambda_2 I)\vec{w} = \vec{0}$

$$\begin{pmatrix} 3-(-1) & 2 & 4 \\ 2 & -(-1) & 2 \\ 4 & 2 & 3-(-1) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 4w_1 + 2w_2 + 4w_3 = 0 \\ 2w_1 + w_2 + 2w_3 = 0 \\ 4w_1 + 2w_2 + 4w_3 = 0 \end{cases} \Rightarrow \begin{array}{l} 2w_1 + w_2 + 2w_3 = 0 \\ w_2 = -2w_1 - 2w_3 \end{array}$$

$$\vec{w} = \begin{pmatrix} w_1 \\ -2w_1 - 2w_3 \\ w_3 \end{pmatrix} = w_1 \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + w_3 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

$\underline{\underline{w_1 = 1, w_3 = 0}}$ $\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = \vec{w_1}$

$\underline{\underline{w_1 = 0, w_3 = 1}}$ $\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} = \vec{w_2}$

correspond to $\lambda = -1$

(c) $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix}$.

$\det(A - \lambda I) = 0$

$$\det(A-\lambda I) = \left| \begin{array}{ccc|cc} 1-\lambda & 0 & 0 & 1-\lambda & 0 \\ 2 & -2-\lambda & 1 & 2 & -2-\lambda \\ 2 & 1 & -2-\lambda & 2 & 1 \end{array} \right.$$

$$= (1-\lambda)(-2-\lambda)(-2-\lambda) - (1-\lambda) = 0$$

$$(1-\lambda)\left[(-2-\lambda)^2 - 1\right] = 0.$$

$$(1-\lambda)(4 + 4\lambda + \lambda^2 - 1) = 0$$

$$(1-\lambda)(\lambda^2 + 4\lambda + 3) = 0.$$

$$(1-\lambda)(\lambda+3)(\lambda+1) = 0.$$

$$\boxed{\lambda_1 = -1, \quad \lambda_2 = -3, \quad \lambda_3 = 1. \quad \text{eigen values}}$$

Eigenvectors: $\lambda_3 = 1$

$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ is a solution of the system

$A - \lambda I = \begin{pmatrix} 1-\lambda & 0 & 0 \\ 2 & -2-\lambda & 1 \\ 2 & 1 & -2-\lambda \end{pmatrix}$

$$(A - \lambda_3 I)\vec{v} = \vec{0}$$

$$\begin{pmatrix} 1-1 & 0 & 0 \\ 2 & -2-1 & 1 \\ 2 & 1 & -2-1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & -3 & 1 \\ 2 & 1 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$$

component form

$$\begin{cases} 2v_1 - 3v_2 + v_3 = 0 \\ 2v_1 + v_2 - 3v_3 = 0 \end{cases}$$

$$-4v_2 - 2v_3 = 0 \Rightarrow \underline{v_3 = -2v_2}$$

$$2v_1 = 3v_3 - v_2$$

$$2v_1 = 3(\underbrace{-2v_2}_{v_3}) - v_2$$

$$2v_1 = -7v_2 \quad \text{or} \quad \underline{v_1 = -\tfrac{7}{2} v_2}$$

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -7/2\, v_2 \\ v_2 \\ -2v_2 \end{pmatrix} \overset{v_2 = 2}{=} \boxed{\begin{pmatrix} -7 \\ 2 \\ -4 \end{pmatrix} \text{ eigenvector corresponding to } \lambda = 1}$$