Brief table of Laplace transform

$f(t) = \mathcal{L}^{-1}\{f\}(s)$	$F(s) = \mathcal{L}\{f\}(s)$
1	$\frac{1}{s}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
$t^n, \quad n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}, \quad s > 0$
$e^{at}t^n, \quad n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
$\sin bt$	$\frac{b}{s^2+b^2}, \quad s > 0$
$\cos bt$	$\frac{s}{s^2+b^2}, \quad s > 0$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
$u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$
$\delta(t-t_0)$	e^{-st_0}
$\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$e^{ct}f(t)$	$F(s-c)$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$

1. Use the definition to find the Laplace transforms of

   (a) $f(t) = e^{at}$ where a is a non zero real number.

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty e^{at}e^{-st}\,dt$$

$$= \int_0^\infty e^{(a-s)t}\,dt = \lim_{N\to\infty} \int_0^N e^{(a-s)t}\,dt$$

$$= \lim_{N\to\infty} \frac{e^{(a-s)t}}{(a-s)}\Bigg|_0^N$$

$$= \lim_{N\to\infty} \frac{e^{(a-s)N}}{(a-s)} - \frac{1}{(a-s)}$$

$s>a$

$(a-s)<0$ $$= 0 - \frac{1}{(a-s)}$$

$$\boxed{\mathcal{L}\{e^{at}\}(s) = \frac{1}{s-a}}$$

(b) $f(t) = \begin{cases} 5-t & 0 \leqslant t < 2 \\ 3t & 2 \leqslant t. \end{cases}$

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t) e^{-st} dt$$

$$= \int_0^2 f(t) e^{-st} dt + \int_2^\infty f(t) e^{-st} dt$$

$$= \int_0^2 (5-t) e^{-st} dt + \int_2^\infty 3t e^{-st} dt$$

$u = 5-t \quad v' = e^{-st} \qquad u = 3t \quad v' = e^{-st}$

$u' = -1 \quad v = \frac{e^{-st}}{-s} \qquad u' = 3 \quad v = \frac{e^{-st}}{-s}$

$$= \left(\left.\frac{(5-t)e^{-st}}{-s}\right|_0^2 - \int_0^2 \frac{e^{-st}}{s} dt \right) + \lim_{N \to \infty} \left.\frac{3te^{-st}}{-s}\right|_2^N + \int_2^N \frac{3e^{-st}}{s} dt$$

$$= \frac{3e^{-2s}}{-s} + \frac{5}{s} + \left.\frac{e^{-st}}{s^2}\right|_0^2$$

$$+ \lim_{N \to \infty} \frac{3Ne^{-sN}}{-s} + \frac{6e^{-2s}}{s} + \left.\frac{3e^{-st}}{-(s^2)}\right|_2^N$$

$s > 0$

$$= \frac{5}{s} - \frac{3e^{-2s}}{s} + \frac{e^{-2s}}{s^2} - \frac{1}{s^2}$$

$$+ \frac{6e^{-2s}}{s} + 0 + \frac{3e^{-2s}}{s^2}$$

$$\boxed{\mathcal{L}\{f(t)\}(s) = \frac{5}{s} - \frac{1}{s^2} + \frac{4e^{-2s}}{s^2} + \frac{3e^{-2s}}{s}}$$

2. Find the inverse Laplace transform of

(a) $F(s) = \dfrac{4}{(s-2)^5}$.

$$\mathcal{L}\{t^4 e^{2t}\} = \frac{4!}{(s-2)^5} = \frac{24}{(s-2)^5}$$

$$F(s) = \frac{4}{(s-2)^5} = \frac{1}{6}\left(\frac{24}{(s-2)^5}\right) = \frac{1}{6}\mathcal{L}\{t^4 e^{2t}\}$$

$$\boxed{\mathcal{L}^{-1}\{F(s)\} = \frac{1}{6} t^4 e^{2t}}$$

2. $F(s) = \frac{8s^2 - 4s + 12}{s(s^2+4)}$. $= \frac{a}{s} + \frac{bs+d}{s^2+4}$

$\times s \quad s=0 \qquad a = \frac{12}{4} = 3$

$\times s \quad s \longrightarrow \infty \qquad \lim_{s\to\infty} \frac{8s^3}{s^3} = 8 = a + b \qquad b = 5$

$s = 1$

$$\frac{8-4+12}{1(1+4)} = \frac{16}{5} = 3 + \frac{5+d}{5}$$

$$16 = 15 + 5 + d$$
$$-4 = d$$

$$F(s) = \frac{3}{s} + \frac{5s-4}{s^2+4}$$

$\mathcal{L}\{\cos 2t\} = \frac{s}{s^2+4}$

$\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$

$$= \mathcal{L}\{3\} + 5\,\frac{s}{s^2+4} - 2\left(\frac{2}{s^2+4}\right)$$

$$= \mathcal{L}\{3 + 5\cos 2t - 2\sin 2t\}$$

$$\boxed{\mathcal{L}^{-1}\{F(s)\} = 3 + 5\cos 2t - 2\sin 2t}$$

3. $F(s) = \dfrac{2s-3}{s^2+2s+10}$. $= \dfrac{2s-3}{s^2+2s+1+9} = \dfrac{2s-3}{(s+1)^2+9}$

$+ \Rightarrow$ irreducible

$\mathcal{L}\{e^{-t}\cos 3t\} = \dfrac{s+1}{(s+1)^2+9}$

$\mathcal{L}\{e^{-t}\sin 3t\} = \dfrac{3}{(s+1)^2+9}$

$$= \frac{2(s+1) - 2 - 3}{(s+1)^2+9}$$

$$= 2\,\frac{(s+1)}{(s+1)^2+9} - \frac{5}{(s+1)^2+9}$$

$$= 2\mathcal{L}\{e^{-t}\cos 3t\} - \frac{5}{3}\mathcal{L}\{e^{-t}\sin 3t\}$$

$$\boxed{\mathcal{L}^{-1}\{F(s)\} = 2e^{-t}\cos 3t - \frac{5}{3}e^{-t}\sin 3t}$$

3. Use the Laplace transform to solve the given initial value problem

(a) $y'' + 3y' + 2y = 4t, \qquad y(0) = 1, \quad y'(0) = 0.$

$$\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = 4\mathcal{L}\{t\}$$

$$s^2\mathcal{L}\{y\} - \underbrace{y'(0)}_{=0} - s\underbrace{y(0)}_{=1} + 3\left(s\mathcal{L}\{y\} - \underbrace{y(0)}_{=1}\right) + 2\mathcal{L}\{y\} = \frac{4}{s^2}$$

$$\underbrace{(s^2+3s+2)}_{\text{characteristic equation}}\mathcal{L}\{y\} - s - 3 = \frac{4}{s^2}$$

$$\mathcal{L}\{y\} = \frac{4 + s^3 + 3s^2}{s^2(s^2+3s+2)} = \frac{4 + s^3 + 3s^2}{s^2(s+1)(s+2)}$$

$$= \frac{a}{s} + \frac{b}{s^2} + \frac{c}{s+1} + \frac{d}{s+2}$$

$\times s^2 \quad s=0 \qquad b = \frac{4}{2} = 2$

$\times(s+1) \quad s=-1 \qquad c = \frac{4-1+3}{1(1)} = 6$

$\times(s+2) \quad s=-2 \qquad d = \frac{4-8+12}{(2)^2(-1)} = \frac{8}{-4} = -2$

$\times s \quad s \to \infty \qquad 1 = a + c + d$

$1 = a + 6 - 2 \qquad a = -3$

$$\mathcal{L}\{y\} = \frac{-3}{s} + \frac{2}{s^2} + \frac{6}{s+1} + \frac{-2}{s+2}$$

$$= \mathcal{L}\{-3 + 2t + 6e^{-t} - 2e^{-2t}\}$$

$$\boxed{y = \underbrace{-3 + 2t}_{y_p} + \underbrace{6e^{-t} - 2 \cdot e^{-2t}}_{y_h}}$$

$y(0) = -3 - 2 + 6 = 1$

$y'(0) = 2 - 6 + 4 = 0$

(b) $y'' + 9y = \cos 2t, \qquad y(0) = 0, \quad y'(0) = 1.$

$$\mathcal{L}\{y''\} + 9\,\mathcal{L}\{y\} = \mathcal{L}\{\cos 2t\}$$

$$s^2\,\mathcal{L}\{y\} - y'(0) - s\,y(0) + 9\,\mathcal{L}\{y\} = \frac{s}{s^2+4}$$

$$(s^2+9)\,\mathcal{L}\{y\} - 1 - 0 \quad = \frac{s}{s^2+4}$$

$$\mathcal{L}\{y\} = \frac{s + s^2 + 4}{(s^2+4)(s^2+9)} = \frac{as+b}{s^2+4} + \frac{cs+d}{s^2+9}$$

$$= \frac{as^3 + 9as + bs^2 + 9b + cs^3 + 4cs + ds^2 + 4d}{(s^2+4)(s^2+9)}$$

$$= \frac{(a+c)s^3 + (b+d)s^2 + (9a+4c)s + 4d + 9b}{(s^2+4)(s^2+9)}$$

coeff s^3 $\quad a + c = 0 \qquad a = -c$

coeff s^2 $\quad b + d = 1$

coeff s $\quad 9a + 4c = 1 \qquad 9a - 4a = 1 \quad 5a = 1 \quad a = \frac{1}{5} \quad c = -\frac{1}{5}$

constant $\quad 4d + 9b = 4$

$$b + d = 1$$
$$9b + 4d = 4$$

$(2) - 4(1) \qquad 5b + 0 = 4 - 4 \quad b = 0 \quad d = 1$

$$\mathcal{L}\{y\} = \frac{\frac{1}{5}s}{s^2+4} + \frac{-\frac{1}{5}s + 1}{s^2+9}$$

$$= \frac{1}{5}\,\mathcal{L}\{\cos 2t\} - \frac{1}{5}\,\mathcal{L}\{\cos 3t\} + \frac{1}{3}\,\mathcal{L}\{\sin 3t\}$$

$$y(t) = \underbrace{\frac{1}{5}\cos 2t}_{y_p} \underbrace{- \frac{1}{5}\cos 3t + \frac{1}{3}\sin 3t}_{y_h}$$

4. Express $f(t)$ in terms of the unit step function $u_c(t)$ and find its Laplace transform.

(a) $f(t) = \begin{cases} t^2, & 0 \leqslant t < 2 \\ e^t, & 2 \leqslant t \end{cases}$.

$f(t) = t^2 + u_2(t) \cdot (e^t - t^2)$

$= t^2 + e^t u_2(t) - t^2 u_2(t)$

$\mathcal{L}\{u_2(t)\} = \dfrac{e^{-2s}}{s}$

$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$, $F(s) = \mathcal{L}\{f\}$

$\mathcal{L}\{e^t u_2(t)\} = \dfrac{e^{-2(s-1)}}{s-1}$

$\mathcal{L}\{t^2 f(t)\} = F''(s)$, $F(s) = \mathcal{L}\{f\}$

$\mathcal{L}\{t^2 u_2(t)\} = \left(\dfrac{e^{-2s}}{s}\right)'' = \left(\dfrac{-2e^{-2s} \cdot s - e^{-2s}}{s^2}\right)'$

$= \left(-\dfrac{2e^{-2s}}{s} - \dfrac{e^{-2s}}{s^2}\right)'$

$= -2 \cdot \dfrac{-2e^{-2s} \cdot s - e^{-2s}}{s^2} - \dfrac{-2e^{-2s} \cdot s^2 - 2s e^{-2s}}{s^4}$

$= \dfrac{4e^{-2s}}{s} + \dfrac{2e^{-2s}}{s^2} + \dfrac{2e^{-2s}}{s^2} + \dfrac{2e^{-2s}}{s^3}$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2\} + \mathcal{L}\{e^t u_2(t)\} - \mathcal{L}\{t^2 u_2(t)\}$

$= \boxed{\dfrac{2}{s^3} + \dfrac{e^{-2(s-1)}}{s-1} - \dfrac{4}{s} e^{-2s} + \dfrac{4e^{-2s}}{s^2} + \dfrac{2e^{-2s}}{s^3}}$

(b) $f(t) = \begin{cases} 2, & 0 \leqslant t < 3 \\ 5t^2, & 3 \leqslant t < 8 \\ 3\cos(t-8), & 8 \leqslant t \end{cases}$ $. = 2 + u_3(t)\left[5t^2 - 2\right] + u_8(t)\left[3\cos(t-8) - 5t^2\right]$

$$= 2 + 5t^2 u_3(t) - 2u_3(t) + 3\cos(t-8)\,u_8(t) - 5t^2 u_8(t)$$

$$\mathcal{L}\{u_3(t)\} = \frac{e^{-3s}}{s}$$

$$\mathcal{L}\{t^2 u_3(t)\} = \left(\frac{e^{-3s}}{s}\right)'' = \left(\frac{-3e^{-3s}\cdot s - e^{-3s}}{s^2}\right)' = \left(-\frac{3e^{-3s}}{s} - \frac{e^{-3s}}{s^2}\right)'$$

$$= -3\cdot\frac{-3e^{-3s}\cdot s - e^{-3s}}{s^2} - \frac{-3e^{-3s}\cdot s^2 - 2se^{-3s}}{s^4}$$

$$= \frac{9e^{-3s}}{s} + \frac{3e^{-3s}}{s^2} + \frac{3e^{-3s}}{s^2} + \frac{2e^{-3s}}{s^3}$$

$$= \frac{9e^{-3s}}{s} + \frac{6e^{-3s}}{s^2} + \frac{2e^{-3s}}{s^3}$$

$$\mathcal{L}\{\cos(t-8)\,u_8(t)\} = \mathcal{L}\{\cos t\}\,e^{-8s} = \frac{s}{s^2+1}e^{-8s}$$

$$\mathcal{L}\{t^2 u_8(t)\} = \left(\frac{e^{-8s}}{s}\right)'' = \frac{64e^{-8s}}{s} + \frac{16e^{-8s}}{s^2} + \frac{2e^{-8s}}{s^3}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{2\} + 5\mathcal{L}\{t^2 u_3(t)\} - 2\mathcal{L}\{u_3(t)\} + 3\mathcal{L}\{\cos(t-8)\,u_8(t)\} - 5\mathcal{L}\{t^2 u_8(t)\}$$

$$= \boxed{\frac{2}{s} + 5\left(\frac{9e^{-3s}}{s} + \frac{6e^{-3s}}{s^2} + \frac{2e^{-3s}}{s^3}\right) - 2\frac{e^{-3s}}{s} + \frac{3s}{s^2+1}e^{-8s} - 5\left(\frac{64e^{-8s}}{s} + \frac{16e^{-8s}}{s^2} + \frac{2e^{-8s}}{s^3}\right)}$$

5. Find the inverse Laplace transform of the given functions

(a) $F(s) = \dfrac{s + 3se^{-5s}}{s^2 - 4s + 3}$

$$= \frac{s}{(s-3)(s-1)} + \left(\frac{3s}{(s-3)(s-1)}\right) e^{-5s}$$

$$\frac{s}{(s-3)(s-1)} = \frac{a}{s-3} + \frac{b}{s-1} = \frac{a(s-1)+b(s-3)}{(s-3)(s-1)} = \frac{3}{2}\left(\frac{1}{s-3}\right) - \frac{1}{2}\left(\frac{1}{s-1}\right)$$

$$= \frac{(a+b)s - (a+3b)}{(s-3)(s-1)} \qquad = \mathcal{L}\left\{\frac{3}{2}e^{3t} - \frac{1}{2}e^{t}\right\}$$

$1 = a + b$ (coeff of s)

$0 = a + 3b$ constant term.

$a = -3b$ $\quad 1 = -3b + b = -2b$ $\quad b = -\frac{1}{2}$ $\quad a = \frac{3}{2}$

$$\mathcal{L}\{\underbrace{u_c(t)\, f(t-c)}\} = \underbrace{e^{-sc}\,\mathcal{L}\{f(t)\}}$$

$$= e^{-5s}\frac{3s}{(s-3)(s-1)} = 3e^{-5s}\,\mathcal{L}\Big\{\underbrace{\frac{3e^{3t}}{2} - \frac{1}{2}e^{t}}_{f(t)}\Big\}$$

$$\frac{3e^{-5t}s}{(s-3)(s-1)} = 3\mathcal{L}\{u_5(t)\, f(t-5)\}$$

$$= 3\mathcal{L}\left\{u_5(t)\left(\frac{3}{2}e^{3(t-5)} - \frac{1}{2}e^{(t-5)}\right)\right\}$$

$$\mathcal{L}^{-1}\{F(s)\} = \left(\frac{3}{2}e^{3t} - \frac{1}{2}e^{t}\right) + 3u_5(t)\left(\frac{3}{2}e^{3(t-5)} - \frac{1}{2}e^{(t-5)}\right)$$

(b) $F(s) = \dfrac{(2s-1)e^{-s}}{s^2-2s+2}$.

$\mathcal{L}\{e^t \cos t\} = \dfrac{s-1}{(s-1)^2+1}$

$\mathcal{L}\{e^t \sin t\} = \dfrac{1}{(s-1)^2+1^2}$

$$\frac{2s-1}{s^2-2s+2} = \frac{2s-1}{(s-1)^2+1} = \frac{2s-2+2-1}{(s-1)^2+1}$$

$$= \frac{2(s-1)}{(s-1)^2+1} + \frac{1}{(s-1)^2+1}$$

$$= 2\mathcal{L}\{e^t \cos t\} + \mathcal{L}\{e^t \sin t\}$$

$$= \mathcal{L}\{2e^t \cos t + e^t \sin t\}.$$

$$F(s) = e^{-s}\mathcal{L}\{2e^t \cos t + e^t \sin t\} = \mathcal{L}\{u_1(t)\,(2e^{(t-1)}\cos(t-1) + e^{(t-1)}\sin(t-1))\}$$

$$\mathcal{L}^{-1}\{F(s)\} = u_1(t)\,(2e^{(t-1)}\cos(t-1) + e^{(t-1)}\sin(t-1))$$

6. Find the solution to the given initial value problem

(a) $y'' + 3y' + 2y = \begin{cases} 1, & 0 \leqslant t < 10 \\ 0, & 10 \leqslant t \end{cases}$, $\quad y(0) = 0, \quad y'(0) = 0.$

(b) $y'' + 4y = u_\pi(t) - u_{3\pi}(t), \quad y(0) = 1, \quad y'(0) = -2$

(c) $y'' + 2y' + 5y = \sin(t) + u_\pi(t)\cos(t - \pi), \quad y(0) = 0, \quad y'(0) = 0.$

Postponed to WIR 8.