ON AMENABLE ITERATED MONODROMY GROUPS

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1. Introduction

Amenability is one of most fundamental finiteness properties in group theory and has many applications in different branches of Mathematics. But even after almost one hundred years of research, the boundary between the classes of amenable and non-amenable groups is not fully understood, and amenability of many groups is still open.

It is clear by now that there is no hope of obtaining a purely algebraic description of the class of amenable groups (e.g., by proving that all amenable groups can be constructed from some "basic" amenable groups using group-theoretic constructions). The property is essentially analytic in its nature. This is also reflected in the fact that amenability of the so-called *non-elementary amenable groups* (roughly speaking, groups that can not be constructed from "obviously" amenable groups) are usually proven using dynamics of their action on topological spaces and properties of associated random walks on the groups or on the orbits of their action.

Our paper is an example of this approach. We prove amenability for a new class of groups of dynamical origin. The proof uses analytic properties of their action, deduces properties of the associated random walks and then uses this information to prove amenability of the group.

Iterated monodromy groups are naturally associated with self-coverings of topological spaces (more generally with pairs of maps $f, \iota : \mathcal{X}_1 \longrightarrow \mathcal{X}$, where f is a covering map). In the case the self-covering is locally expanding and the space is compact, the iterated monodromy group (together with an additional purely algebraic structure) is a complete invariant of the map up to topological conjugacy. The relation between the iterated monodromy group and an expanding map is very close and one can be effectively used to study the other. See [Nek05, Nek14] for the general theory and applications.

It was shown in [Nek10] that iterated monodromy groups of expanding maps (more generally, *contracting self-similar groups*) can not have free non-abelian subgroups. Consequently, there are no obvious reasons why they are not amenable. In fact, no example of a non-amenable iterated monodromy group of an expanding self-covering of a compact space is known.

Another class of groups for which amenability is an open questions are groups generated by automata (transducers) of *polynomial activity growth*, defined originally in [Sid00]. It was proved in [Sid04] that such groups can not have free subgroups. Amenability of some groups in this class was proved in [BKN10, AAV13, JNdlS16]. The general case remains to be open.

Both questions (amenability of iterated monodromy groups and of groups generated by automata of polynomial activity growth) remain to be out of reach of the current methods of proving amenability.

The goal of our paper is to show that every group in the intersection of these two classes is amenable. This provides a new interesting class of amenable groups. Perhaps more importantly, we show a relation of amenability to analytic and topological properties of dynamical systems.

The second section of the paper is an overview of the main definitions of the theory of self-similar groups. Section 3 defines automata of polynomial activity growth and gives a criterion when they generate a contracting self-similar group. Iterated monodromy groups are defined in Section 4. The next section describes how the iterated monodromy groups can be used to approximate the corresponding expanding self-covering by simpler topological spaces (e.g., by graphs).

Section 6 describes the methods of proving amenability of groups developed in [JNdlS16], and how they can be applied to groups generated by automata of polynomially growing activity.

The last two sections contain the main results of the paper. In Section 7, we prove that the graphs of the canonical action on the Cantor set of the iterated monodromy group of a post-critically finite rational function is always recurrent (i.e., that the simple random walk on it is recurrent). The proof is based on the fact that these graphs are sub-graphs with *Speiser graphs* associated with the leaves of the inverse limit of backward iterations of the function (known as the *Lyubich-Minski lamination*). It was proved by Lyubich and Minski in [LM97] that the corresponding leaves are parabolic. This implies, by results of [Doy84] and [Mer08] that the graphs are recurrent. The latter is a crucial ingredient which can be used to prove amenability of iterated monodromy groups generated by automata of polynomial activity growth using the methods of [JNdlS16]. The results of the last section of our paper supersedes this result, but the fact that the orbital graphs of iterated monodromy groups of rational functions are recurrent is interesting by itself, so we decided to keep it, especially since the proof is a nice combination of classical results.

In the last section of the paper we prove that any contracting self-similar group generated by automata of polynomial activity growth is amenable. The proof is similar to the main result of Section 7, but instead of using conformal geometry of surfaces, we use more general techniques of Ahlfors-regular conformal dimension. Namely, we prove that the associated *limit space* of the self-similar group has conformal dimension 1, which implies that the orbital graphs are recurrent. This approach was inspired by the papers [Thu16, Thu19].

2. Self-similar groups

We give here an overview of the basic definitions. For more, see [Nek05, Nek11, Nek08].

2.1. **Bisets.** Let G be a discrete group. A *G-biset* is a set $\mathfrak{B}$ together with commuting left and right actions of G. A *self-similar group* is a pair $(G, \mathfrak{B})$, where $\mathfrak{B}$ is a G-biset such that the right action has finitely many orbits and is free (the latter means that $x \cdot g = x$ for $x \in \mathfrak{B}$ and $g \in G$ implies $g = 1$).

If $\mathfrak{B}_1$ is a right G-set, and $\mathfrak{B}_2$ is a left G-set, then we define $\mathfrak{B}_1 \otimes_G \mathfrak{B}_2 = \mathfrak{B}_1 \otimes \mathfrak{B}_2$ as the quotient of $\mathfrak{B}_1 \times \mathfrak{B}_2$ by the equivalence relation $(x_1, x_2) = (x_1 \cdot g, g^{-1} \cdot x_2)$, $g \in G$. If $\mathfrak{B}_1$ is a right H-set for some group H such that the H-action and the G-action on $\mathfrak{B}_1$ commute, then $\mathfrak{B}_1 \otimes \mathfrak{B}_2$ is naturally a left H-set. Similarly, if $\mathfrak{B}_2$ is a right H-set such that the H-action and the left G-action commute, then $\mathfrak{B}_1\otimes\mathfrak{B}_2$ is a right H-set.

In particular, if $\mathfrak{B}_1$, $\mathfrak{B}_2$ are G-bisets, then we have the biset $\mathfrak{B}_1\otimes\mathfrak{B}_2$. It is easy to prove that $(\mathfrak{B}_1\otimes\mathfrak{B}_2)\otimes\mathfrak{B}_3$ and $\mathfrak{B}_1\otimes(\mathfrak{B}_2\otimes\mathfrak{B}_3)$ are isomorphic as bisets, so that for every biset $\mathfrak{B}$ we have well defined bisets $\mathfrak{B}^{\otimes n}$. We define $\mathfrak{B}^{\otimes 0}$ to be the group G itself with the natural left and right G-actions. We denote the biset $\bigsqcup_{n=0}^\infty\mathfrak{B}^{\otimes n}$ by $\mathfrak{B}^*$.

The set of right-orbits $\mathfrak{B}^*/G$ has a natural structure of a rooted tree with the root equal to the unique G-orbit on $\mathfrak{B}^{\otimes 0}=G$. A vertex $vG\in\mathfrak{B}^{\otimes n}/G$ is connected to the vertices of the form $(v\otimes x)G\in\mathfrak{B}^{\otimes(n+1)}/G$ for $x\in\mathfrak{B}$. It is not hard to check that this is a regular rooted d-tree, where d is the cardinality of the set $\mathfrak{B}/G$ of the right G-orbits in $\mathfrak{B}$.

The left action defines a left action of G by automorphisms on the tree $\mathfrak{B}^*/G$. We say that the self-similar group G is *faithful* if this action is faithful. In general, the kernel of the action of G on $\mathfrak{B}^*/G$ is the maximal normal subgroup N of G such that for every $h\in N$ and $x\in\mathfrak{B}$ there exists $h'\in N$ such that $h\cdot x=x\cdot h'$. Then the *faithful quotient* of $(G,\mathfrak{B})$ is the self-similar group $(G/N,\mathfrak{B}/N)$, where $\mathfrak{B}/N$ is the set of right N-orbits.

It is easy to check that if $(G,\mathfrak{B})$ is a self-similar group, then $(G,\mathfrak{B}^{\otimes n})$ is also a self-similar group.

2.2. **Automata.** Let $(G,\mathfrak{B})$ be a self-similar group, and let $\mathsf{X}\subset\mathfrak{B}$ be a basis of $\mathfrak{B}$ (i.e., a right orbit transversal). We will denote by X^n the set of elements $x_1\otimes x_2\otimes\cdots\otimes x_n\in\mathfrak{B}^{\otimes n}$, and write these elements just as $x_1x_2\ldots x_n$. It is checked directly that X^n is a basis of $\mathfrak{B}^{\otimes n}$. In other words, every right G-orbit in $\mathfrak{B}^*$ has a unique element of the set $\mathsf{X}^*=\bigcup_{n=0}^\infty\mathsf{X}^n$. We get a bisection between the tree $\mathfrak{B}^*/G$ and the tree of X^*. A word $v\in\mathsf{X}^*$ is connected in the tree X^* to the words of the form vx for $x\in\mathsf{X}$. The root of the tree X^* is the empty word corresponding to the root of $\mathfrak{B}^*/G$.

After identification of the tree $\mathfrak{B}^*/G$ with X^* we get the corresponding left action of G on X^* coming from the left action on $\mathfrak{B}^*/G$. We call it the *standard action* of G on X^*.

For every $x\in\mathsf{X}$ and $g\in G$ there exist unique $y\in\mathsf{X}$ and $h\in G$ such that $g\cdot x=y\cdot h$. We denote $y=g(x)$ and $h=g|_x$. Then the standard action of G on X^* is given by the recursive rule

$$g(xw)=g(x)g|_x(w).$$

More generally, if $v\in\mathsf{X}^n$ and $g\in G$, then there exists a unique element $g|_v\in G$ such that $g\cdot v=g(v)\cdot g|_v$ in $\mathfrak{B}^{\otimes n}$. We have then

$$g(vw)=g(v)g|_v(w)$$

for all $v,w\in\mathsf{X}^*$ and $g\in G$. We also have

$$(g_1g_2)|_v=g_1|_{g_2(v)}g_2|_v,\qquad g|_{v_1v_2}=g|_{v_1}|_{v_2}.$$

We interpret the above recursive rule as the work of an automaton with the set of states G which reading a letter x on the input gives $g(x)$ to the output, and changes its state to $g|_x$. This automaton, if g is its initial state, will process a word v letter by letter, and give $g(v)$ as the output. Note that this automaton depends on the choice of the basis $\mathsf{X}\subset\mathfrak{B}$.

The *Moore diagram* of this automaton is the graph with the set of states G in which for every $g \in G$ and $x \in \mathsf{X}$ there is an arrow starting in g, ending in $g|_x$, and labeled by $(x, g(x))$. In order to find $g(x_1 x_2 \dots x_n)$ one has to find the unique directed path starting in g and labeled by $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$. Then $g(x_1 x_2 \dots x_n) = y_1 y_2 \dots y_n$.

For a given $g \in G$ we can consider the subgraph consisting of all vertices reachable from g, i.e., the subgraph spanned by $\{g|_v \ : \ v \in \mathsf{X}^*\}$. It enough to know this subgraph in order to compute the action of g. We call it the *automaton defining* g.

In general, an *(invertible) automaton* is a collection $\mathcal{A}$ of automorphisms of the rooted tree X^* such that for every $g \in \mathcal{A}$ and $x \in \mathsf{X}$ there exists $g|_x \in \mathcal{A}$ such that $g(xw) = g(x)g|_x(w)$ for all $w \in \mathsf{X}^*$. The *Moore diagram* of $\mathcal{A}$ is the graph with the set of vertices $\mathcal{A}$ in which for every $g \in \mathcal{A}$ and $x \in \mathsf{X}$ there is an arrow from g to $g|_x$ labeled by $(x, g(x))$.

If $\mathcal{A}$ is an automaton, then the group of automorphisms of X^* generated by $\mathcal{A}$ is self-similar. The corresponding biset $\mathfrak{B}$ is the set of transformations of X^* of the form $v \mapsto xg(v)$ for $x \in \mathsf{X}$ and $g \in G$. The left and right actions of G are given by the post- and pre-compositions with the elements of G. More explicitly, if we write the transformation $v \mapsto xg(v)$ just as $x \cdot g$, then the biset operations are given by

$$g_1 \cdot (x \cdot g) \cdot g_2 = g_1(x) \cdot g_1|_x g g_2.$$

The standard action on X^* of this self-similar group coincides with the original action of G on X^*.

2.3. **Contracting groups.** Let $(G, \mathfrak{B})$ be a self-similar group, and let $\mathsf{X} \subset \mathfrak{B}$ be a basis. We say that the standard action of G on X^* is *contracting* if there exists a finite set $\mathcal{N} \subset G$ such that for every $g \in G$ there exists n such that $g|_v \in \mathcal{N}$ for all words $v \in \mathsf{X}^*$ of length at least n. The smallest set $\mathcal{N}$ with this property is called the *nucleus* of the action.

The nucleus is an automaton, i.e., for every $g \in \mathcal{N}$ and $x \in \mathsf{X}$ we have $g|_x \in \mathcal{N}$. Accordingly, the *Moore diagram* of the nucleus is the graph with the set of vertices $\mathcal{N}$ in which for every $g \in \mathcal{N}$ and $x \in \mathsf{X}$ there is an arrow from g to $g|_x$ labeled by $(x, g(x)$.

It is proved in [Nek05, Corollary 2.11.7] that if the standard action is contracting for one basis, then it is contracting for every basis of the biset, i.e., the property of being contracting depends only on the self-similar group $(G, \mathfrak{B})$. We say then that the biset $\mathfrak{B}$ is *hyperbolic.*

Note that if the action of G on X^* is contracting, then the set of states $\{g|_v \ : \ v \in \mathsf{X}^*\}$ of the automaton defining g is finite.

Let $(G, \mathfrak{B})$ be a contracting self-similar group and $\mathsf{X} \subset \mathfrak{B}$ is a basis. Consider the space $\mathsf{X}^{-\omega}$ of left-infinite sequences $\dots x_2 x_1$ with the direct product topology. We say that $\dots x_2 x_1, \dots y_2 y_1$ are *asymptotically equivalent* if there exists a finite subset $N \subset G$ and a sequence $g_k \in N$ such that $g_k(x_k \dots x_2 x_1) = y_k \dots y_2 y_1$ for every $k \geq 1$. One can show, see [Nek05, Proposition 3.2.6], that $\dots x_2 x_1$ and $\dots y_2 y_1$ are asymptotically equivalent if and only if there exists a directed path $\dots e_2 e_1$ of in the Moore diagram of the nucleus such that the arrow e_i is labeled by (x_i, y_i).

The quotient of $\mathsf{X}^{-\omega}$ by the asymptotic equivalence relation is called the *limit space* of $(G, \mathfrak{B})$. We denote it by $\mathcal{J}_G$. The shift $\dots x_2 x_1 \mapsto \dots x_3 x_2$ induces a finite-to-one continuous map $\mathcal{J}_G \longrightarrow \mathcal{J}_G$, which we call the *limit dynamical system* of the self-similar group. It does not depend, up to a topological conjugacy, on the choice of the basis.

Similarly, consider the space $\mathsf{X}^{-\omega} \times G$, where G is discrete. We say that two points $\ldots x_2x_1 \cdot g$ and $\ldots y_2y_1 \cdot h$ are *asymptotically equivalent* if there exists a finite set B and a sequence $g_k \in B$ such that $g_k \cdot x_k \ldots x_2x_1 \cdot g = y_k \ldots y_2y_1 \cdot h$ in $\mathfrak{B}^{\otimes k}$. Again, one can show that one can chose $g_k \in \mathcal{N}$ such that $g_k \cdot x_k = y_k \cdot g_{k-1}$ and $g_0g = h$. In other words the points are asymptotically equivalent if and only if there exists a directed path $\ldots e_2e_1$ in the Moore diagram of the nucleus ending in hg^{-1} and such that e_i is labeled by (x_i, y_i). Denote by $\mathcal{X}_G$ the quotient of $\mathsf{X}^{-\omega} \times G$ by the asymptotic equivalence relation.

The natural right action of G on $\mathsf{X}^{-\omega} \times G$ preserves the asymptotic equivalence classes, hence $\mathcal{X}_G$ is a right G-space. Moreover, there is a natural G-equivariant homeomorphism $\mathcal{X}_G \otimes \mathfrak{B} \longrightarrow \mathcal{X}_G$ induced by $(\ldots x_2x_1 \cdot g) \otimes (x \cdot h) = \ldots x_2x_1g(x) \cdot g|_x h$ for $x_i, x \in \mathsf{X}$ and $g, h \in G$.

One can define the limit G-space $\mathcal{X}_G$ without choosing a basis of the biset in the following way. For every finite subset $A \subset \mathfrak{B}$ consider the space of sequences $A^{-\omega}$ with the direct product topology, and then pass to the direct limit of the topological spaces $A^{-\omega}$ with respect to the natural inclusions $A_1^{-\omega} \subset A_2^{-\omega}$ for all finite subsets $A_1 \subset A_2$. Let Ω be the obtained topological space. Then declare two sequences $(\ldots, a_2, a_1)$ and $(\ldots, b_2, b_1)$ equivalent if there exists a finite set $B \subset G$ and a sequence $g_k \in B$ such that $g_k \cdot a_k \otimes \cdots \otimes a_2 \otimes a_1 = b_k \otimes \cdots \otimes b_2 \otimes b_1$. The quotient of Ω by this equivalence relation is the limit G-space $\mathcal{X}_G$. So, in some sense, $\mathcal{X}_G$ is the "bounded" infinite tensor power $\mathfrak{B}^{\otimes(-\omega)}$.

The right action of G on $\mathcal{X}_G$ is proper and co-compact, and $\mathcal{J}_G$ is naturally homeomorphic to the quotient space $\mathcal{X}_G/G$. This shows that, in fact, it is more natural to consider $\mathcal{J}_G$ as an *orbispace*, since the action of G on $\mathcal{X}_G$ is not free.

2.4. **Orbital graphs and graphs of germs.** Let G be a group generated by a finite set S and acting on a set X. The corresponding *graph of the action* is the graph with the set of vertices X in which for every $x \in X$ and $s \in S$ there is a directed edge starting in x and ending in $s(x)$. We often label this edge by s. If the action is transitive, then for every $x_0 \in X$ the graph of the action is naturally isomorphic to the *Schreier graph* of G modulo the stabilizer G_{x_0} of x_0 in G. Namely, the points of X are in a bijective correspondence with the cosets hG_{x_0}, and the vertex corresponding to a coset hG_{x_0} is connected to the vertex corresponding to the coset shG_{x_0} for every $h \in G$ and $s \in S$.

In general, the *orbital graph* is the graph of the action of G on an orbit of the action. The orbital graph Γ_x of a point $x \in X$ is therefore naturally isomorphic to the Schreier graph of G modulo the stabilizer G_x.

Suppose that X is a topological space. A *germ* (g, x) is the equivalence class of the pair (g, x) with respect to the equivalence relation identifying (g_1, x) and (g_2, x) if there exists a neighborhood U of x such that $g_1|_U = g_2|_U$.

The *graph of germs* $\widetilde{\Gamma}_x$ is the graph with the set of vertices equal to the set of germs (g, x) for $g \in G$, in which for every germ (g, x) and generator $s \in S$ we have an arrow from (g, x) to the germ (sg, x) labeled by s. Note that the map $(g, x) \mapsto g(x)$ is a surjective covering map from the graph of germs $\widetilde{\Gamma}_x$ to the orbital graph Γ_x.

The graph of germs $\widetilde{\Gamma}_x$ is naturally isomorphic to the Schreier graph of G modulo the subgroup $G_{(x)}$ of elements of g acting identically on a neighborhood of x. It is

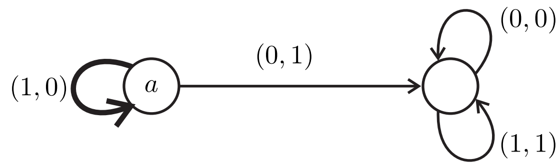


FIGURE 1. The adding machine

easy to see that $G_{(x)}$ is a normal subgroup of the stabilizer G_x. In particular, the covering map $\widetilde{\Gamma}_x \longrightarrow \Gamma_x$ is regular (Galois) with the group of deck transformations $G_x/G_{(x)}$. The group $G_x/G_{(x)}$ is called the *group of germs* at the point x.

3. AUTOMATA OF POLYNOMIALLY GROWING ACTIVITY

3.1. **Cyclic structure.** Let g be an automorphism of the tree X^*. Consider the following function

$$\alpha_g(n) = |\{v \in \mathsf{X}^n \ : \ g|_v \neq 1\}|.$$

In other words, $\alpha_g(n)$ counts the number of paths of length n in the Moore diagram of g starting in g and ending in a non-trivial state. Let Γ_g be the Moore diagram of g in which we delete the trivial state and all the arrows adjacent (i.e., incoming or outgoing) to it. Then $\alpha_g(n)$ is the number of paths of length n in Γ_g starting in g. If A is the adjacency matrix for the graph Γ_g, then $\alpha_g(n)$ is the sum of entries in the column corresponding to the vertex g in the matrix A^n. It follows that if g is finite state, then the formal series $\sum_{n=0}^{\infty} \alpha_g(n)t^n$ is a rational function. In particular, $\alpha_g(n)$ grows either polynomially or exponentially. We have the following description of the growth of $\alpha_g(n)$. See a proof in [Sid00].

Proposition 3.1. *Let g be a finite-state automorphism of X^*. The function $\alpha_g(n)$ grows exponentially if and only if there are two directed cycles in Γ_g with a common vertex.*

Otherwise $\alpha_g(n)$ is bounded from above by a polynomial. Then the smallest degree of a polynomial bounding $\alpha_g(n)$ is d if and only if $d+1$ is the maximal length of sequence $C_1, C_2, \ldots, C_{d+1}$ of pairwise different cycles of Γ_g such that for every $i = 1, 2, \ldots, d$ there is a directed path from a vertex of C_i to a vertex of C_{i+1}.

We have

$$\alpha_{g_1 g_2}(n) \leq \alpha_{g_1}(n) + \alpha_{g_2}(n),$$

since $(g_1 g_2)|_v \neq 1$ implies that either $g_2|_v \neq 1$ or $g_1|_{g_2(v)} \neq 1$. It follows that the set of automorphisms g of X^* such that $\alpha_g(n)$ is bounded by a polynomial of degree d is a subgroup of the group of automorphisms of X^*. Denote by $\mathcal{P}_d(\alpha)$ the intersection of this group with the group of finite-state automorphisms of X^*.

Example 3.2. Consider the automaton over the alphabet $\mathsf{X} = \{0, 1\}$ given by

$$a(0w) = 1w, \qquad a(1w) = 0a(w).$$

The Moore diagram of the automaton is shown on Figure 1. The transformation defined by a is called the (binary) *adding machine* or *odometer*. It follows from the structure of the automaton that it belongs to $\mathcal{P}_0(\mathsf{X})$. (We have highlighted the non-trivial cycle in the Moore diagram.)

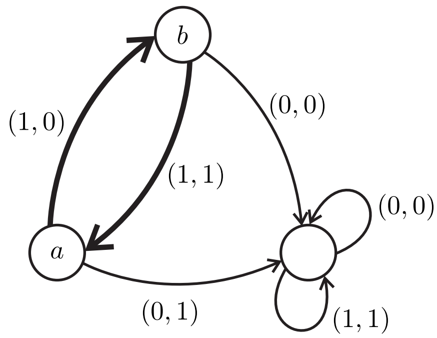


Figure 2. The iterated monodromy group of $z^2 - 1$

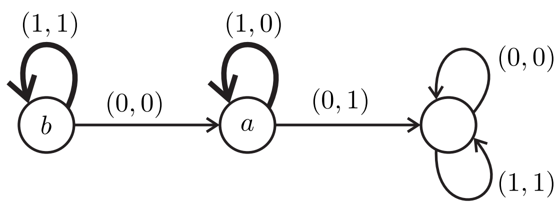


Figure 3. Automaton in $\mathcal{P}_1(\mathsf{X})$

Example 3.3. Another example of a subgroup of $\mathcal{P}_0(\mathsf{X})$ is generated by the transformations

$$a(0w) = 1w, \qquad a(1w) = 0b(w),$$
$$b(0w) = 0w, \qquad b(1w) = 1a(w).$$

The corresponding Moore diagram is shown on Figure 2. The group $\langle a, b\rangle$ is the *iterated monodromy group* of the complex polynomial $z^2 - 1$, see below. In fact, the iterated monodromy groups of post-critically finite polynomials are subgroups of $\mathcal{P}_0(\mathsf{X})$. The group $\langle a, b\rangle$ appeared (even before the iterated monodromy groups were introduced) in [GŻ02].

Example 3.4. Consider the transformations

$$a(0w) = 1w, \qquad a(1w) = 0a(w),$$
$$b(0w) = 0a(w), \qquad b(1w) = 1b(w),$$

see the Moore diagram on Figure 3. We see that $\langle a, b\rangle < \mathcal{P}_1(\mathsf{X})$. The graphs of the action of this group on the boundary of the tree were defined for the first time in [BH05]. These graphs where studied later as orbital graphs of the group $\langle a, b\rangle$ in [BCSDN12].

3.2. **Contracting subgroups of $\mathcal{P}_d(\mathsf{X})$.** Let $G < \mathcal{P}_d(\mathsf{X})$ be a finitely generated self-similar group. Let $S = S^{-1}$ be a finite self-similar generating set. We consider S as an automaton. Let N be a number divisible by the lengths of all cycles in the Moore diagram of S. Let C be the set of words $v \in \mathsf{X}^N$ such that there exists $g \in S$ such that $g|_v = g$. In other words, C is the set of words of length N that are read on cycles of the Moore diagram of C.

Consider the graph Γ with the set of vertices C, where for every $g \in S$ and $v \in C$ such that $g|_v = g$, we have an arrow from v to $g(v)$ labeled by g. Note that then $g(v) \in C$ and there is an arrow from $g(v)$ to v labeled by g^{-1}. Note also that if $g_k, g_{k-1}, \ldots, g_1$ are labels of arrows in a direct path from a vertex v_1 to a vertex $v_2 \in C$, then we have $g|_{v_1} = g$ and $g(v_1) = v_2$ for $g = g_1 g_2 \cdots g_k$. Consider the set $\mathcal{G}$ of pairs (v, g), where $v \in C$, and g is a product of labels in Γ in a path from v to $g(v)$. Then $\mathcal{G}$ is a groupoid: if $(g_1, v_1), (g_2, v_2) \in \mathcal{G}$ are such that $g_2(v_2) = v_1$, then (taking concatenation of the corresponding paths) we see that $(g_1 g_2, v_2) \in \mathcal{G}$ and (taking inverse of the path) that $(g_1^{-1}, g_1(v_1)) \in \mathcal{G}$.

Proposition 3.5. *Let $G < \mathcal{P}_d(\mathsf{X})$ be a finitely generated self-similar group. Then the following conditions are equivalent.*

1. *The groupoid $\mathcal{G}$ is finite.*
2. *The isotropy groups of $\mathcal{G}$ are finite.*
3. *The groups of germs of the action of G on X^ω are finite.*
4. *The group G is contracting.*

Proof. The groupoid $\mathcal{G}$ has finitely many units (the vertices of the graph Γ). Hence, it is finite if and only if its isotropy groups are finite, so that we have (1)$\Longleftrightarrow$(2).

Let us show (1)$\Longrightarrow$(4). Suppose that the groupoid $\mathcal{G}$ is finite. We want to prove that G is contracting. The proof essentially is the same as the proof that all self-similar subgroups of $\mathcal{P}_0(\mathsf{X})$ are contracting, see [BN03] and [Nek05, Theorem 3.9.12]. Let $g \in G$, and let us write g as a product $g_1 g_2 \cdots g_n$ of elements of S. Let $w = x_1 x_2 \ldots \in \mathsf{X}^\omega$, and consider the sequence of the sections

$$g|_{x_1 x_2 \ldots x_k} = (g_1 g_2 \cdots g_n)|_{x_1 x_2 \ldots x_k} = g_1|_{y_{1,1} y_{1,2} \ldots y_{1,k})} g_2|_{y_{2,1} y_{2,2} \ldots y_{2,k})} \cdots g_n|_{y_{n,1} y_{n,2} \ldots y_{n,k}},$$

where $y_{i,1} y_{i,2} \ldots = g_{i+1} g_{i+2} \cdots g_n(x_1 x_2 \ldots)$ and $y_{n,1} y_{n,2} \ldots = x_1 x_2 \ldots$. For every i, the sequence $g_i|_{y_{i,1} y_{i,2} \ldots y_{i,k}}$, $k = 1, 2, \ldots$, is a path in the Moore diagram of S. It follows from the structure of automata of polynomial growth that this sequence will be eventually contained in a cycle of S or will be eventually trivial. It follows that there exists k_0 such that for every $i = 1, 2, \ldots, n$ the sequence $g_i|_{y_{i,1} y_{i,2} \ldots y_{i,k}}$ is either trivial or belongs to a cycle of the Moore diagram of S for all $k \geq k_0$. Then, for $v_i = y_{i,k_0+1} y_{i,k_0+2} \ldots y_{i,k_0+N}$ and $h_i = g_i|_{y_{i,1} y_{i,2} \ldots y_{i,k}}$, we have

$$h_i(v_i) = v_{i-1}, \quad h_i|_{v_i} = h_i$$

for all $i = 1, 2, \ldots, n$ (the condition $h_i(v_i) = v_{i-1}$ is void for $i = 1$). It follows that $(h_1 h_2 \cdots h_n, v_n)$ belongs to the groupoid $\mathcal{G}$. We have proved that for every $g \in G$ and $x_1 x_2 \ldots \in \mathsf{X}^\omega$ there exists k_0 such that $g|_{x_1 x_2 \ldots x_{k_0}}$ belong to the finite set N of elements $h \in G$ such that $(h, v) \in \mathcal{G}$ for some v. It follows that the set of all sections of the elements of N is finite and is the nucleus of G.

If G is contracting, then groups of germs of its action on X^ω are finite, since their size is bounded from above by the size of the nucleus, i.e., (4)$\Longrightarrow$(3).

It remains to show that (3)$\Longrightarrow$(2). Suppose that $G < \mathcal{P}_d(\mathsf{X})$ is a finitely generated self-similar group such that an isotropy group of $\mathcal{G}$ is infinite. Let us show that there exists a point $w \in \mathsf{X}^\omega$ such that the group of germs $G_{(w)}$ is infinite. Suppose that the isotropy group of a point $v \in C$ in the groupoid $\mathcal{G}$ is infinite. The isotropy group $\mathcal{G}_v$ consists of elements $g \in G$ such that $g|_v = g$ and $g(v) = v$. Then $g(v^\omega) = v^\omega$ for every $g \in \mathcal{G}_v$, and the action of every non-trivial element $g \in \mathcal{G}_v$ is non-trivial on every neighborhood of v^ω, since $g|_{v^n} = g$ for every n. It follows that the group $\mathcal{G}_v$ is faithfully represented in the group of germs at w, hence the group of germs is infinite. $\square$

Let C be a compact countable metrizable space. Define $C_0 = C$, and inductively, for a non-limit ordinal $\alpha+1$, $C_{\alpha+1}$ to be the complement of the set of isolated points of C_α. If α is a limit ordinal, then define $C_\alpha = \bigcap_{\beta<\alpha} C_\beta$. Then every C_α is a compact countable metrizable space. Note also that $C_{\alpha+1}$ is a proper subset of C_α. It follows that there exists a countable ordinal α such that $C_\alpha = \emptyset$. In fact, α can not be a limit ordinal, by compactness, so that it is equal to $\beta+1$ for some β. Let us call β the *Cantor-Bendixson rank* of C. Note that C_β is a finite set. It is known that the pair $(\beta, |C_\beta|)$ is a complete invariant of C up to homeomorphism (namely, by a theorem of S. Mazurkiewicz and W. Sierpiński [MS20] C is homeomorphic to the ordinal $\omega^\beta \cdot n + 1$ with the order topology).

Proposition 3.6. *Let G be a contracting group. Let $\mathcal{X}_G$ be the limit G-space, and let $\mathcal{T} \subset \mathcal{X}_G$ be the corresponding tile (i.e., the image of $\mathsf{X}^{-\omega} \cdot 1$ in $\mathcal{X}_G$. Then $G \le \mathcal{P}_d(\mathsf{X})$ for some d if and only if the boundary $\partial\mathcal{T}$ is countable. Moreover, if d is the smallest number such that $G \le \mathcal{P}_d(\mathsf{X})$, then the Cantor-Bendixson rank of ∂T is equal to d.*

Proof. It is known that $\partial\mathcal{T}$ is the image of the set of sequences $\ldots x_2x_1$ such that there exists a path in the nucleus of G ending in a non-trivial state and labeled by $\ldots(x_2, y_2), (x_1, y_1)$ for some $y_i \in \mathsf{X}$. Two sequences $\ldots x_2x_1$ and $\ldots y_2y_1$ represent the same point of $\mathcal{X}_G$ if and only if there exists a path labeled by $\ldots(x_2, y_2)(x_1, y_1)$ and ending in the trivial state. Let $\mathcal{S}$ be the space of paths (seen as a subset of the space of sequences of edges) in the nucleus of the group ending in a non-trivial state. It is easy to see that $\mathcal{S}$ is a compact countable space if the $G \le \mathcal{P}_d(\mathsf{X})$. Otherwise, the nucleus has two intersecting non-trivial cycles, which implies that $\mathcal{S}$ is uncountable. Let $\pi : \mathcal{S} \longrightarrow \partial\mathcal{T}$ map a path $\ldots, e_2, e_1$ labeled by $\ldots(x_2, y_2), (x_1, y_1)$ to $\ldots x_2x_1$. Note that if we know $\ldots x_2x_1$ and the origin of the edge e_n, then we know all the edges e_i for $i \le n$. It follows that the cardinality of $\pi^{-1}(\xi)$ is bounded from above by the size of the nucleus. In particular, since π is surjective, ∂T is countable if and only if $\mathcal{S}$ is countable.

Suppose that $f : C_1 \longrightarrow C_2$ is a finite-to-one surjective map, where C_1 and C_2 are compact countable metrizable spaces. If a point $x \in C_2$ is isolated, then $f^{-1}(x)$ is open and finite, hence consists of isolated points. On the other hand, if all points of $f^{-1}(x)$ are isolated, then $C_1 \setminus f^{-1}(x)$ is compact, hence its image $C_2 \setminus \{x\}$ is closed, i.e., x is isolated. Consequently, if C_1' is the set of non-isolated points of C_1, then $f(C_1')$ is equal to the set of non-isolated points of C_2. The map $f : C_1' \longrightarrow f(C_1')$ is a finite-to-one sujrective map. It follows that if C_1 has finite Cantor-Bendixson rank, then the Cantor-Bendixson rank of C_1 is equal to the Cantor-Bendixson rank of C_2.

Let us order the non-trivial cycles of the Moore diagram of the nucleus of G by saying that a cycle is greater than another cycle if there exists a directed path from the former to the latter. If d is smallest such that $G \le \mathcal{P}_d(\mathsf{X})$, then $d+1$ is the length of the longest strictly decreasing sequence of cycles.

It is easy to see that a path $(\ldots, e_2, e_1) \in \mathcal{S}$ is isolated in $\mathcal{S}$ if and only if all but possibly a finite number of edges e_i belong to a cycle maximal in the above defined order. After we remove the isolated points of $\mathcal{S}$ we get the space $\mathcal{S}_1$ of all infinite sequences $(\dots, e_2, e_1)$ inside the automaton A_1 consisting of the non-maximal cycles of the nucleus and their states. A point of $\mathcal{S}_1$ is then isolated if and only if all but finitely many of its edges belong to a maximal cycle of A_1. It follows then inductively that the Cantor-Bendixson rank of $\mathcal{S}$ is equal to the length of the longest strictly decreasing sequences of non-trivial cycles in the nucleus minus one. □

Example 3.7. If $(G, \mathfrak{B})$ is a self-similar group, then the for different bases X of $\mathfrak{B}$ the standard action of G on X^* may belong to different groups $\mathcal{P}_d(\mathsf{X})$, or not belong to any of them. For example, the iterated monodromy group of $z^2 - 1$ can be defined as the group generated by

$$a(0w) = 1w, \qquad a(1w) = 0b(w),$$
$$b(0w) = 0w, \qquad b(1w) = 1a(w).$$

This group of automorphisms of the tree X^* is contained in $\mathcal{P}_0(\mathsf{X})$. If we pass to the basis $0' = 0, 1' = 1 \cdot a$, then the recursive definition becomes

$$a(0'w) = 1'a^{-1}(w), \qquad a(1'w) = 0'ba(w),$$
$$b(0'w) = 0'w, \qquad b(1'w) = 1'a(w).$$

Note that since

$$ba(0'w) = 1'w, \qquad ba(1'w) = 0'ba(w),$$

this action is a subgroup of $\mathcal{P}_1(\mathsf{X})$.

If we change the basis to $0'' = 0a, 1'' = 1b$, we get

$$a(0''w) = 1''b^{-1}a(w), \qquad a(1''w) = 0''a^{-1}b^2(w),$$
$$b(0''w) = 0''w, \qquad b(1''w) = 1''b^{-1}ab(w).$$

This group is not contain in $\mathcal{P}_d(\mathsf{X})$ for any d. For example, we have the following two intersecting cycles of states of the automaton:

$$a|_{0'} = a^{-1}b^2, \quad (a^{-1}b^2)|_{1''} = ab, \quad (ab)|_{1''} = a^{-1}bab, \quad (a^{-1}bab)|_{0''} = a,$$

and

$$(b^{-1}a)|_{1''} = a^{-1}b^2, \quad (a^{-1}b^2)|_{1''} = ab, \quad (ab)|_{0''} = b^{-1}a.$$

4. Iterated monodromy groups

4.1. **Main definitions.** A *topological virtual endomorphism* is a pair $f, \iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$, where $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is a finite degree covering, and ι is a continuous map. In particular, a *partial self-covering* is a virtual endomorphism in which ι is an embedding, i.e., it is a covering map of $\mathcal{M}$ by a subset of $\mathcal{M}$.

Suppose that $\mathcal{M}$ is path-connected. Choose a basepoint $t \in \mathcal{M}$. Then the *associated* $\pi_1(\mathcal{M}, t)$*-biset* $\mathfrak{B}_{f,\iota,t}$ is the set of pairs $([\ell], z)$, where $z \in f^{-1}(t)$, and $[\ell]$ is the homotopy class of a path from t to $\iota(z)$.

The right $\pi_1(\mathcal{M}, t)$-action is given by appending the loops $([\ell], z) \cdot [\gamma] = ([\ell\gamma], z)$. (We multiply here the loops in the "unnatural": in a path $\ell\gamma$ the path γ is traversed before the path ℓ.) It is easy to see that the right action is free and that two elements $([\ell_i], z_i)$ belong to the same right orbit if and only if $z_1 = z_2$.

The left $\pi_1(\mathcal{M}, t)$-action is defined by lifting loops using f, and then mapping them back to $\mathcal{M}$ using ι. Namely, for $([\ell], z) \in \mathfrak{B}_{f,\iota,t}$ and $\gamma \in \pi_1(\mathcal{M}, t)$, there is a unique lift γ_z of γ by f that starts at z. Let z' be the end of γ. Then we define $[\gamma]\cdot([\ell], z) = ([\iota(\gamma_z)\ell, z')$. (Note that we need to record z in the definition of elements of $\mathfrak{B}_{f,\iota,t}$ only in the case that ι is not injective on $f^{-1}(t)$.)

Topological virtual endomorphisms can be naturally iterated. Namely, denote by $\mathcal{M}_n$ the set of all sequences $(x_1, x_2, \ldots, x_n) \in \mathcal{M}_1^n$ such that $f(x_i) = \iota(x_{i+1})$. Then the pair of maps $f^n(x_1, x_2, \ldots, x_n) := f(x_n)$ and $\iota^n(x_1, x_2, \ldots, x_n) := \iota(x_1)$ is a virtual endomorphism from $\mathcal{M}_n$ to $\mathcal{M}$, which we call the *nth iteration* of the virtual endomorphism $f, \iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$. One can show that the $\pi_1(\mathcal{M}, t)$-biset associated with $f^n, \iota^n : \mathcal{M}_n \longrightarrow \mathcal{M}$ is naturally isomorphic to $\mathfrak{B}_{f,\iota,t}^{\otimes n}$.

Definition 4.1. The *iterated monodromy group* $\mathrm{IMG}(f, \iota)$ of $f, \iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is the faithful quotient of the self-similar group $(\pi_1(\mathcal{M}, t), \mathfrak{B}_{f,\iota,t})$.

The tree $\mathfrak{B}_{f,\iota,t}^{\infty}/\pi_1(\mathcal{M}, t)$ on which the iterated monodromy group acts is naturally isomorphic to the *tree of preimages* T_t, which is defined as the set of points $(x_1, x_2, \ldots, x_n) \in \mathcal{M}_n$ such that $f(x_n) = t$, with $t \in \mathcal{M}_0$ as the root, and a vertex $(x_1, x_2, \ldots, x_n) \in \mathcal{M}_n$ connected to the vertex $(x_2, x_3, \ldots, x_n) \in \mathcal{M}_{n-1}$. The fundamental group $\pi_1(\mathcal{M}, t)$ acts on T_t by lifting loops by the covering maps $f^n : \mathcal{M}_n \longrightarrow \mathcal{M}$. Namely, for every vertex $z = (x_1, x_2, \ldots, x_n)$ of T_t and every loop $\gamma \in \pi_1(\mathcal{M}, t)$ there is a unique lift of γ by f^n starting in z. The end of this lift is a vertex of T_t which is the image of z by the action of γ.

The iterated monodromy group is the group of the automorphisms of the tree T_t defined by the elements of the fundamental group.

In the case of a partial self-covering (i.e., when ι is the identical embedding of a subset $\mathcal{M}_1$ to $\mathcal{M}$), map $(x_1, x_2, \ldots, x_n) \mapsto x_1$ is a homeomorphism of $\mathcal{M}_n$ with the domain of the nth iteration f^n of the partial self-covering. Then the set of vertices of the tree of preimages T_t is just the disjoint union of the sets $f^{-n}(t)$.

A basis of $\mathfrak{B}_{f,\iota,t}$ is, by definition, a choice of homotopy classes $[\ell_z]$ of paths connecting t to $\iota(z)$ for each $z \in f^{-1}(t)$. We will usually choose an alphabet X of cardinality equal to the degree of f, a bijection $\Lambda : \mathsf{X} \longrightarrow f^{-1}(t)$, and a collection of paths ℓ_x connecting t to $\iota(\Lambda(x))$. Then the corresponding standard action of the fundamental group (and of the iterated monodromy group) is described as follows (the proof is just a direct application of the definition of $\mathfrak{B}_{f,\iota,t}$).

Proposition 4.2. *For $\gamma \in \pi_1(\mathcal{M}, t)$ and $x \in \mathsf{X}$, denote by γ_x the lift of γ by f starting in $\Lambda(x)$, and denote by $\gamma(x) \in \mathsf{X}$ the letter such that $\Lambda(\gamma(x))$ is the end of γ_x. Then the recurrent rules*

$$\gamma(xv) = \gamma(x)(\ell_{\gamma(x)}^{-1}\iota(\gamma_x)\ell_x)(v)$$

define the standard action of $\pi_1(\mathcal{M}, t)$ on X^* conjugate to the iterated monodromy action of $\pi_1(\mathcal{M}, t)$ on the tree of preimages T_t.

4.2. **Complex rational functions.** A rational function $f \in \mathbb{C}(z)$ is said to be *post-critically finite* if the forward orbit of every critical point of f is finite. The union of the orbits of the *critical values* is called the *post-critical set* of f and is denoted P_f.

If f is post-critically finite, then $f : \widehat{\mathbb{C}} \setminus f^{-1}(P_f) \longrightarrow \widehat{\mathbb{C}} \setminus P_f$, where $\widehat{\mathbb{C}}$ is the Riemann sphere, is a partial self-covering of a finitely punctured sphere. The *iterated monodromy group* $\mathrm{IMG}(f)$ of f is, by definition, the iterated monodromy group of this partial self-covering.

The iterated monodromy groups of rational functions can be computed using Proposition 4.2. For instance, the iterated monodromy group of $z^2 - 1$ is given in

Example 3.7. We will see more examples of iterated monodromy groups of rational functions below.

4.3. **Natural extension.** Let $f, \iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$ be a topological virtual endomorphism, and let $\mathcal{M}_n$ be as above. Denote by $\widehat{\mathcal{M}}$ the inverse limit of the sequence

$$\mathcal{M} \xleftarrow{f} \mathcal{M}_1 \xleftarrow{f_1} \mathcal{M}_2 \xleftarrow{f_2} \cdots,$$

where $f_n : \mathcal{M}_{n+1} \longrightarrow \mathcal{M}_n$ is the map $(x_0, x_1, \ldots, x_n) \mapsto (x_1, x_2, \ldots, x_n)$.

The space $\widehat{\mathcal{M}}$ is naturally identified with the space of sequences $(\ldots, x_2, x_1) \in \mathcal{M}_1^{-\omega}$ such that $f(x_i) = \iota(x_{i+1})$.

The subset of $\widehat{\mathcal{M}}$ consisting of points $(\ldots, x_2, x_1)$ such that $f(x_1) = t$ is naturally identified with the boundary ∂T_t of the tree of preimages. In particular, the iterated monodromy group $\mathrm{IMG}(f, \iota)$ acts on it by homeomorphisms.

The next lemma follows directly from the definitions.

Lemma 4.3. *Let $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ be a partial self-covering. Two points $x, y \in \partial T_t$ belong to one path connected component of $\widehat{\mathcal{M}}$ if and only if x and y belong to one orbit of the iterated monodromy group action.*

We get then the following description of the orbital graphs of the action of the iterated monodromy group.

Corollary 4.4. *If S is a set of loops generating $\mathrm{IMG}(f, \iota)$ and let Γ_0 be the graph equal to the union of the elements of S. Then the orbital graphs with respect to S of the action of $\mathrm{IMG}(f, \iota)$ on the boundary of the tree of preimages are isomorphic to the preimages of Γ_0 in the path-connected components of $\widehat{\mathcal{M}}$.*

5. Contracting models

The first two subsection is an overview of [Nek14].

5.1. **Basic definitions and results.** Let $(G, \mathfrak{B})$ be a self-similar group. Let $\mathcal{X}$ be a locally compact metric space with a right proper and co-compact G-action. Then $\mathcal{X} \otimes \mathfrak{B}$ is also a right G-space. If $\mathcal{X}$ is a graph and G acts on it by automorphisms, then $\mathcal{X} \otimes \mathfrak{B}$ is also a graph.

We are interested in G-equivariant maps $I : \mathcal{X} \otimes \mathfrak{B} \longrightarrow \mathcal{X}$. In order to define I, one has to define the map $I(\cdot \otimes x) : \mathcal{X} \longrightarrow \mathcal{X}$ for every $x \in \mathsf{X}$. These maps have to satisfy the conditions $I(t \cdot g \otimes x) = I(t \otimes g \cdot x)$ and $I(t \otimes x \cdot g) = I(t \otimes x) \cdot g$. In particular, if we choose a basis X of $\mathfrak{B}$, then it is enough to define $I(\cdot \otimes x)$ for $x \in \mathsf{X}$ subject to the condition

$$I(t \cdot g \otimes x) = I(t \otimes g(x)) \cdot g|_x.$$

for all $g \in G$ and $x \in \mathsf{X}$. Any collection of maps $I(\cdot \otimes x) : \mathcal{X} \longrightarrow \mathcal{X}$ satisfying the above condition defines a G-equivariant map $I : \mathcal{X} \otimes \mathfrak{B} \longrightarrow \mathcal{X}$.

The map I can be used then to define G-equivariant maps $I_n^m : \mathcal{X} \otimes \mathfrak{B}^{\otimes m} \longrightarrow \mathfrak{B}^{\otimes n}$ by the rule

$$I_n^m(t \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_m) = I(\ldots I(I(t \otimes x_1) \otimes x_2) \ldots \otimes x_{m-n}) \otimes x_{m-n+1} \otimes \cdots \otimes x_m,$$

for all $m > n \geq 0$. We have $I_k^n \circ I_n^m = I_k^m$ for all $m > n > k$.

Lemma 5.1. *Suppose that $\mathcal{X}$ is path-connected, and the action of G on $\mathcal{X}$ is free. Let $\mathcal{M} = \mathcal{X}/G$ and $\mathcal{M}_1 = (\mathcal{X} \otimes \mathfrak{B})/G$ be the corresponding spaces of orbits. Then the correspondence $\xi \otimes x \mapsto \xi$ from $\mathcal{X} \otimes \mathfrak{B}$ to $\mathcal{X}$ induces a well defined covering map $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$.*

Proof. The map f is well defined, since $\xi_1 \otimes x$ and $\xi_2 \otimes y$ belong to one G-orbit only if ξ_1 and ξ_2 belong to the same G-orbit.

Since the action of G on $\mathcal{X}$ is free and proper, for every $\xi \in \mathcal{X}$ there exists a neighborhood U of ξ such that for every $g \in G$ either $g = 1$ or $U \cdot g \cap U = \emptyset$. The images of $U \otimes x$ and $U \otimes y$ in $\mathcal{M}_1$ intersect if and only if there exists $g \in G$ and $\xi_1, \xi_2 \in U$ such that $\xi_1 \otimes x = \xi_2 \otimes y \cdot g$. But the latter means that there exists $h \in G$ such that $\xi_1 \cdot g = \xi_2$ and $g^{-1} \cdot x = y \cdot g$. The first equality implies $g = 1$ and $\xi_1 = \xi_2$. We see that the images $U \otimes x$ and $U \otimes y$ in $\mathcal{M}$ intersect if and only if they coincide. Consequently, the image of U in $\mathcal{M}$ is evenly covered by f. □

If $I : \mathcal{X} \otimes \mathfrak{B} \longrightarrow \mathcal{X}$ is G-equivariant, then it induces a continuous map $\iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$. We see that every G-equivariant map I naturally defines a topological virtual endomorphism $f, \iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$ for $\mathcal{M} = \mathcal{X}/G$ and $\mathcal{M}_1 = \mathcal{X} \otimes \mathfrak{B}/G$.

Let us show that the iterated monodromy group $\mathrm{IMG}(f, \iota)$ of the constructed topological virtual endomorphism coincides with the faithful quotient of $(G, \mathfrak{B})$. Let let t be a basepoint of $\mathcal{M}$ equal to the orbit of a point $\xi \in \mathcal{X}$. If γ is a loop in $\mathcal{M}$ based at t, then its lift to $\mathcal{X}$ starting in ξ will end in $\xi \cdot \phi(\gamma)$ for some $\phi(\gamma) \in G$. The map $\phi : \pi_1(\mathcal{M}, t) \longrightarrow G$ will be a well defined epimorphism. Elements of $\mathfrak{B}_{f,\iota,t}$ are pairs $([\ell], z)$, where z is the orbit of a point $\xi \otimes x$ for some $x \in \mathfrak{B}$, and ℓ is a path from t to $\iota(z)$. The lift of ℓ to $\mathcal{X}$ starting in ξ will end in a point in the G-orbit of $I(\xi \otimes x)$. Note that the points $I(\xi \otimes x \cdot g) = I(\xi \otimes x) \cdot g$ are different for different g. Consequently, there is a unique element $x \in \mathfrak{B}$ such that the lift of ℓ starting in ξ ends in $I(\xi \otimes x)$. We get a surjective map $\Phi : ([\ell], z) \mapsto x$ from $\mathfrak{B}_{f,\iota,t}$ to $\mathfrak{B}$. It is checked then directly that Φ and ϕ preserve the biset structures, which will imply that the standard actions of G and $\pi_1(\mathcal{M}, t)$ on the respective rooted trees will coincide.

We see that finding a G-space $\mathcal{X}$ and an equivariant map $I : \mathcal{X} \otimes \mathfrak{B} \longrightarrow \mathcal{X}$ is essentially equivalent to realizing G as the iterated monodromy group of a topological correspondence.

The case when the action of G on $\mathcal{X}$ is non-free can be also included, if we naturally extend the notion of a topological virtual endomorphism to *orbispaces*. We will not do it in this paper, but just use an essentially equivalent (and only slightly less elegant) approach of proper G-spaces and G equivariant maps instead.

5.2. **Contracting virtual endomorphisms.** As we have seen the maps $I_n^m : \mathcal{X} \otimes \mathfrak{B}^{\otimes m} \longrightarrow \mathcal{X} \otimes \mathfrak{B}^{\otimes n}$ satisfy $I_k^n \circ I_n^m = I_k^m$. In particular, we can pass to the inverse limit of the spaces $\mathcal{X} \otimes \mathfrak{B}^{\otimes n}$ with respect to the maps I_n^m.

The following theorem is proved in [Nek14]. It shows that if $I : \mathcal{X} \otimes \mathfrak{B} \longrightarrow \mathcal{X}$ is contracting, then the G-spaces $\mathcal{X} \otimes \mathfrak{B}^{\otimes n}$ converge to the limit G-space $\mathcal{X}_G$. In particular, $\mathcal{X}_G$ can be defined as the unique proper co-compact G-space such that there exists a G-equivariant contracting homeomorphism $\mathcal{X}_G \otimes \mathfrak{B} \longrightarrow \mathcal{X}_G$. In terms of topological correspondences, it says that if $f, \iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is such that f is a local isometry and ι is locally contracting, then $\mathcal{M}_n$ converge to the limit space $\mathcal{J}_G$ of the iterated monodromy group $G = \mathrm{IMG}(f, \iota)$, so that $f_n : \mathcal{M}_{n+1} \longrightarrow \mathcal{M}_n$

converge to the limit dynamical system and $\iota : \mathcal{M}_{n+1} \longrightarrow \mathcal{M}_n$ converge to the identity homeomorphism.

Theorem 5.2. *Let $I : \mathcal{X} \otimes \mathfrak{B} \longrightarrow \mathcal{X}$ be a G-equivariant map such that there exists $\lambda \in (0,1)$ such that for all $t_1, t_2 \in \mathcal{X}$ and $x \in \mathfrak{B}$ we have $|I(t_1 \otimes x) - I(t_2 \otimes x)| \le \lambda |t_1 - t_2|$. Then the inverse limit of the G-spaces $\mathcal{X} \otimes \mathfrak{B}^{\otimes n}$ is G-equivariantly homeomorphic to the limit G-space $\mathcal{X}_G$.*

Note that it is enough to check the contraction condition for x belonging to a basis of $\mathfrak{B}$.

The limits of the maps I_n^m are G-equivariant maps $I_n^\infty : \mathcal{X}_G \longrightarrow \mathcal{X} \otimes \mathfrak{B}^{\otimes n}$.

It is shown in [Nek14] that for every contracting group $(G, \mathfrak{B})$ there exists $n \ge 1$, an affine simplicial complex $\mathcal{X}$, and a piecewise affine map $I : \mathcal{X} \otimes \mathfrak{B}^{\otimes n} \longrightarrow \mathcal{X}$ satisfying the conditions of Theorem 5.2.

5.3. **p-contracting maps.** Let $(G, \mathfrak{B})$ be a self-similar group, and let $\mathsf{X} \subset \mathfrak{B}$ be a basis of the biset. Let Γ be a connected locally finite metric graph (i.e., a locally finite one-dimensional CW-complex such that every edge (1-cell) is isometric to $[0, l]$ for some $l > 0$). We will denote the length of a path γ in Γ by $l(\gamma)$. Let G act on it from the right by a proper co-compact isometric action.

Suppose that $I : \Gamma \otimes \mathfrak{B} \longrightarrow \Gamma$ is such that the maps $I(\cdot \otimes x) : \Gamma \longrightarrow \Gamma$ are piecewise linear. We say that I is *p-contracting* if there exists $\lambda \in (0,1)$ such that for almost all points t of Γ we have

$$\lambda := \operatorname*{ess\,sup}_{t \in \Gamma} \sum_{x \in \mathsf{X}} \left| \frac{dI(t \otimes x)}{dt} \right|^p < 1.$$

If we denote $I_x(t) = I(t \otimes x)$, then we have $I_0^n(t \otimes x_1 x_2 \dots x_n) = I_{x_n}(\dots I_{x_2}(I_{x_1}(t)))$, so that

$$\frac{dI_0^n(t \otimes x_1 x_2 \dots x_n)}{dt} = \left. \frac{dI_0^{n-1}(s \otimes x_2 x_3 \dots x_n)}{ds} \right|_{s = I(t \otimes x_1)} \frac{dI(t \otimes x_1)}{dt}.$$

Consequently,

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t \in \Gamma} \sum_{v \in \mathsf{X}^n} \left| \frac{dI_0^n(t \otimes v)}{dt} \right|^p = \\
&\quad \operatorname*{ess\,sup}_{t \in \Gamma} \sum_{x \in \mathsf{X}} \sum_{v \in \mathsf{X}^{n-1}} \left| \left. \frac{dI_0^{n-1}(s \otimes v)}{ds} \right|_{s = I(t \otimes x)} \right|^p \cdot \left| \frac{dI(t \otimes x)}{dt} \right|^p \le \\
&\quad \operatorname*{ess\,sup}_{t \in \Gamma} \sum_{x \in \mathsf{X}} \left(\operatorname*{ess\,sup}_{s \in \Gamma} \sum_{v \in \mathsf{X}^{n-1}} \left| \frac{dI_0^{n-1}(s \otimes v)}{ds} \right|^p \right) \cdot \left| \frac{dI(t \otimes x)}{dt} \right|^p = \\
&\quad \left(\operatorname*{ess\,sup}_{s \in \Gamma} \sum_{v \in \mathsf{X}^{n-1}} \left| \frac{dI_0^{n-1}(s \otimes v)}{ds} \right|^p \right) \cdot \left(\operatorname*{ess\,sup}_{t \in \Gamma} \sum_{x \in \mathsf{X}} \left| \frac{dI(t \otimes x)}{dt} \right|^p \right) = \\
&\qquad \lambda \left(\operatorname*{ess\,sup}_{s \in \Gamma} \sum_{v \in \mathsf{X}^{n-1}} \left| \frac{dI_0^{n-1}(s \otimes v)}{ds} \right|^p \right),
\end{aligned}$$

hence for every $n \ge 1$ we have

$$\operatorname*{ess\,sup}_{t \in \Gamma} \sum_{v \in \mathsf{X}^n} \left| \frac{dI_0^n(t \otimes v)}{dt} \right|^p \le \lambda^n.$$

We will denote by $l(\gamma)$ the length of a path γ.

Proposition 5.3. *Suppose that $I : \Gamma \otimes \mathfrak{B} \longrightarrow \Gamma$ is p-contracting, and let λ be as in the definition. Then for every path γ in Γ we have*

$$\sum_{v \in \mathsf{X}^n} l(I_0^n(\gamma \otimes v))^p \leq \lambda^n l(\gamma)^p$$

for every n.

Proof. Let $\gamma : [0, L] \longrightarrow \Gamma$ be a geodesically parametrized path in Γ. Then for every $v \in \mathsf{X}^n$ the path $I_0^n(\gamma \otimes v)$ has length $\int_0^L \left| \frac{dI_0^n(\gamma(t) \otimes v)}{dt} \right| \, dt$. We have, by Jensen's inequality,

$$\left(\frac{1}{L} \int_0^L \left| \frac{dI_0^n(\gamma(t) \otimes v)}{dt} \right| \, dt \right)^p \leq \frac{1}{L} \int_0^L \left| \frac{dI_0^n(\gamma(t) \otimes v)}{dt} \right|^p \, dt.$$

Consequently,

$$\sum_{v \in \mathsf{X}^n} (l(I_0^n(\gamma \otimes v))^p \leq L^{p-1} \int_0^L \sum_{v \in \mathsf{X}^n} \left| \frac{dI_0^n(\gamma(t) \otimes v)}{dt} \right|^p \, dt \leq L^p \lambda^n,$$

which finishes the proof. $\square$

6. Amenability of groups

6.1. **Definitions.** A discrete group G is said to be *amenable* if there exists a map $\mu : 2^G \longrightarrow [0, 1]$ such that $\mu(A \cup B) = \mu(A) + \mu(B)$ for any two disjoint subsets $A, B \subset G$, $\mu(G) = 1$, and $\mu(gA) = \mu(A)$ for all $A \subset G$ and $g \in G$.

By a theorem of Tarsky, a group G is amenable if and only if it does not admit a *paradoxical decomposition*, i.e., a decomposition into a disjoint union $G = A_1 \sqcup A_2 \sqcup \ldots \sqcup A_n \sqcup B_1 \sqcup B_2 \sqcup \ldots \sqcup B_m$ for which there exist $g_1, g_2, \ldots, g_n, h_1, h_2, \ldots, h_m \in G$ such that $G = g_1 A_1 \sqcup g_2 A_2 \sqcup \ldots \sqcup g_n A_n = h_1 B_1 \sqcup h_2 B_2 \sqcup \ldots \sqcup h_m B_m$.

Amenability was introduced by J. von Neumann in [vN29] in his analysis of the Hausdorff-Banach-Tarsky paradox. He showed that the class of amenable groups is closed under passing to a subgroup and quotients, under group extensions, and direct unions of groups. Finite groups are obviously amenable. It is also not hard to show that $\mathbb{Z}$ is amenable. In general, a group is called *elementary amenable* if it can be constructed from finite groups and $\mathbb{Z}$ by the above mentioned operations (applied transfinitely). On the other hand, it is also not hard to show that free non-abelian groups are not amenable. Consequently, every group containing a free subgroup is not amenable.

For a long while these where the only two classes for which amenability or non-amenability was known. The first example of an amenable group not belonging to the class of elementary amenable groups is the Grigorchuk group of intermediate growth [Gri80]. The first examples of non-amenable groups not containing free subgroups are infinite Burnside groups [Ol′80].

The border between the class of amenable and non-amenable groups is still not very well understood. The most notorious example is the Thompson group [Tho80, CFP96], amenability of which remains to be a famous open problem. There are many other examples of groups and classes of groups amenability of which is open.

For example, it is not known if all contracting self-similar groups are amenable. Similarly, it is not known if the groups $\mathcal{P}_d(\mathsf{X})$ are amenable for all d and $|\mathsf{X}|$. It is known that $\mathcal{P}_0(\mathsf{X})$ and $\mathcal{P}_1(\mathsf{X})$ are amenable, see [BKN10] and [AAV13], respectively. (See also [JNdlS16], for a more general approach to both results.)

Note that it is known that neither contracting groups nor the groups $\mathcal{P}_d(\mathsf{X})$ contain free subgroups, see [Sid04] and [Nek10]. It is also known for many groups in these classes that they do not belong to the class of elementary amenable groups [Jus18].

6.2. **Amenability of topological full groups.** Amenability of many non-elementary amenable groups can be proved using the techniques of *extensively amenable actions* [JM13, JNdlS16, JMMdlS18].

Definition 6.1. Let G be a group acting on a topological space $\mathcal{X}$. The *(topological) full group* of the action is the group of homeomorphisms $f : \mathcal{X} \longrightarrow \mathcal{X}$ such that for every $x \in \mathcal{X}$ there exists a neighborhood U of x and an element $g \in G$ such that $g|_U = f|_U$. In other words, a homeomorphism belongs to the full group if it locally belongs to G.

Example 6.2. Consider the Cantor set X^ω of infinite sequences over a finite alphabet X. Denote by G_n the group of homeomorphisms of X^ω isomorphic to the symmetric group S_{X^n} of all permutations of X^n acting naturally on X^ω by the action on the prefixes. We obviously have $G_n \le G_{n+1}$ for every n. Denote by S_{X^∞} the direct union of the groups G_n. Note that S_{X^∞} is a topological full group (of its own action), i.e., every homeomorphism locally belonging to S_{X^∞} belongs to it. Note also that S_{X^∞} is amenable, since it is locally finite.

Example 6.3. The intersection of the group of automorphisms of the tree X^* with the group from the previous example is called the *group of finitary automorphisms* of the tree X^*. It consists of automorphisms g for which there exists n such that $g|_v = 1$ for all words $v \in \mathsf{X}^*$ of length at least n (equivalently, for all $v \in \mathsf{X}^n$). This group is isomorphic to the inductive limit of the iterated permutational wreath products $S_{\mathsf{X}} \wr S_{\mathsf{X}} \wr \cdots \wr S_{\mathsf{X}}$. The group of all automorphisms of X^* is isomorphic to the projective limit of these finite groups.

The following theorem from [JNdlS16] gives a method of extending amenable full groups.

Theorem 6.4. *Let H be a group of homeomorphisms of a compact space $\mathcal{X}$ such that the topological full group of the action is amenable. Let G be a finitely generated group of homeomorphisms of $\mathcal{X}$. Suppose that the following conditions are satisfied.*

(1) *For every $g \in G$ there exists a finite set of points $\Sigma_g \subset \mathcal{X}$ such that for every $y \in \mathcal{X} \setminus \Sigma_g$ there exists $h \in H$ and a neighborhood U of y such that $h|_U = g|_U$.*
(2) *The orbital graphs of the action $(G, \mathcal{X})$ are recurrent.*
(3) *The groups of germs of the action $(G, \mathcal{X})$ are amenable.*

The the group G and its topological full group are amenable.

Here a graph is said to be *recurrent* if the simple random walk on it is recurrent, i.e., returns to the origin with probability 1. Note that it is enough to check condition (1) for generators of G.

Theorem 6.5. *Let $G < \mathcal{P}_d(\mathsf{X})$ be a finitely generated self-similar group. If the orbits of G on X^ω are recurrent, then G is amenable.*

It is an open question if the condition of recurrence of the orbits can be dropped. It was proved in [AAV13] that the orbits are always recurrent for $d \leq 1$ (see also [JNdlS16]). It was shown in [AV14], however, that there are examples of subgroups of $\mathcal{P}_d(\mathsf{X})$ with transient orbits for every $d \geq 3$.

Proof. The proof is by induction on the degree d, where we include the base case of "negative one" degree corresponding to the group of finitary automorphisms of X^*, which is amenable as a locally finite group.

Let (G, X) be a self-similar group. Then the full group of the action of (G, X^ω) consists of all homeomorphisms g of X^ω for which there exists n, a permutation $\sigma \in S_{\mathsf{X}^n}$, and a collection $g_v \in G$, $v \in \mathsf{X}^n$, such that $g(x_1x_2\ldots) = \sigma(x_1x_2\ldots x_n)g_{x_1x_2\ldots x_n}(x_{n+1}x_{n+2}\ldots)$ for all $x_1x_2\ldots \in \mathsf{X}^\omega$. In other words, the full group is the natural direct union of the groups $G^{\mathsf{X}^n} \ltimes S_{\mathsf{X}^n}$. In particular, the full group of the action (G, X^ω) is amenable if and only if G is amenable.

Suppose that we know that the theorem is true for all degrees less than d. Let G be as in the theorem. Let G be a group satisfying the conditions of the theorem. Let $\mathcal{N}$ be its nucleus, and let H be the group generated by $\mathcal{N} \cap \mathcal{P}_{d-1}(\mathsf{X})$.

The orbital graphs of the action (H, X^ω) are subgraphs of the orbital graphs of (G, X^ω), therefore they are also recurrent (see [Woe00, Corollary 2.15]). Since $H \leq \mathcal{P}_{d-1}(\mathsf{X})$, this implies that H is amenable, by the inductive hypothesis. Consequently, the topological full group of the action (H, X^ω) is amenable.

It follows from the structure of automata in $\mathcal{P}_d(\mathsf{X})$ that for every $g \in \mathcal{N}$ there exists a finite set of points $\Sigma_g \subset \mathsf{X}^\omega$ (the infinite sequences read along the cycles of states not belonging to $\mathcal{P}_{d-1}(\mathsf{X})$) such that for every $x_1x_2\ldots \notin \Sigma_g$ there exists n such that $g|_{x_1x_2\ldots x_n} \in \mathcal{P}_{d-1}(\mathsf{X})$. It follows that condition (1) of Theorem 6.4 is satisfied. Condition (2) is a part of the assumption.

It remains to prove condition (3). Suppose that $g \in G$ fixes a point $w = x_1x_2\ldots$ of X^ω. There exists n such that $g|_{x_1x_2\ldots x_n} \in \mathcal{N}$. The germ $(g, x_1x_2\ldots)$ is uniquely determined by w, n, and $g|_{x_1x_2\ldots x_n}$. In particular, the group of germs $G_{(w)}$ has not more than $|\mathcal{N}|$ elements. Consequently, there exists n and a group $N \subset \mathcal{N}$ isomorphic to $G_{(w)}$. Namely, every non-trivial element $h \in N$ fixes $x_{n+1}x_{n+2}\ldots$, all sections $h|_{x_{n+1}x_{n+2}\ldots x_{n+m}}$ are non-trivial, and for every $g \in G$ fixing w there exists m and $h \in N$ such that $g|_{x_1x_2\ldots x_{n+m}} = h|_{x_{n+1}x_{n+2}\ldots x_{n+m}}$. The map $g \mapsto h$ induces then the isomorphism of $G_{(w)}$ with N.

Note that it follows from the fact that the sections $h|_{x_{n+1}x_{n+2}\ldots x_{n+m}}$ are non-trivial that the sequences $x_{n+1}x_{n+2}\ldots$ and $(h|_{x_{n+1}x_{n+2}\ldots x_{n+m}})_{m\geq 1}$ are eventually periodic, so by changing n we may assume that they are periodic. If k is the period, then every $h \in N$ is uniquely determined by the permutations defined by $h, h|_{x_{n+1}}, h_{x_{n+1}x_{n+2}}, \ldots, h_{x_{n+1}x_{n+2}\ldots x_{n+k-1}}$ on X and by the sections $h|_{x_{n+1}x_{n+2}\ldots x_{n+i}y}$ for $y \in \mathsf{X} \setminus x_{n+i+1}$. Note that the sections belong to $\mathcal{P}_{d-1}(\mathsf{X})$. It follows that N is isomorphic to a subgroup of the kth direct power of wreath products of H with symmetric groups $S_{|\mathsf{X}|-1}$. As we assume that H is amenable, this implies that the groups of germs of G are amenable. $\square$

7. Rational functions with $\mathrm{IMG}(f) < \mathcal{P}_d(\mathsf{X})$

Theorem 7.1. *Let f be a post-critically finite rational function. Then the orbital graphs of the action of* $\mathrm{IMG}\, f$ *on X^ω are recurrent.*

Proof. Consider the inverse limit of the sequence $\widehat{\mathbb{C}} \xleftarrow{f} \widehat{\mathbb{C}} \xleftarrow{f} \widehat{\mathbb{C}} \xleftarrow{f} \cdots$ and remove from it the points $(x_0, x_1, x_2, \ldots)$ containing infinitely many critical points (i.e., the sequences contained in the superattracting cycles of f). Let $\mathcal{L}$ be the corresponding space. It is the *Lyubich-Minsky lamination* studied in [LM97].

The space $\mathcal{L}$ contains as an open subset the inverse limit $\widehat{\mathcal{X}}$ of the iterations of the partial self-covering $f : \widehat{\mathbb{C}} \setminus f^{-1}(P_f) \longrightarrow \widehat{\mathbb{C}} \setminus P_f$. The difference $\mathcal{L} \setminus \widehat{\mathcal{X}}$ is the set of sequences $(x_0, x_1, x_2, \ldots) \in \mathcal{L}$ such that $x_0 \in P_f$, since P_f is forward-invariant. Since P_f is finite, there exists a neighborhood U of x_0 homeomorphic to a disc and not containing any other point of P_f. Since f is a local homeomorphism at all but finitely many points x_i, and acts in some local charts at x_i as z^k acts at zero for the remaining x_i, the full preimage $\widehat{U}$ of U in $\mathcal{L}$ will be homeomorphic to a direct product $U_1 \times C$ of a disc U_1 with a Cantor set C such that the projection map $\widehat{U} \longrightarrow U$ is of the form $(t, \xi) \mapsto (\phi(t), x_0)$, where $\phi : U_1 \longrightarrow U$ is a finite degree branched covering. It follows that two points belong to the same path-connected component of $\widehat{\mathcal{X}}$ if and only if they belong to the same path-connected component of $\mathcal{L}$.

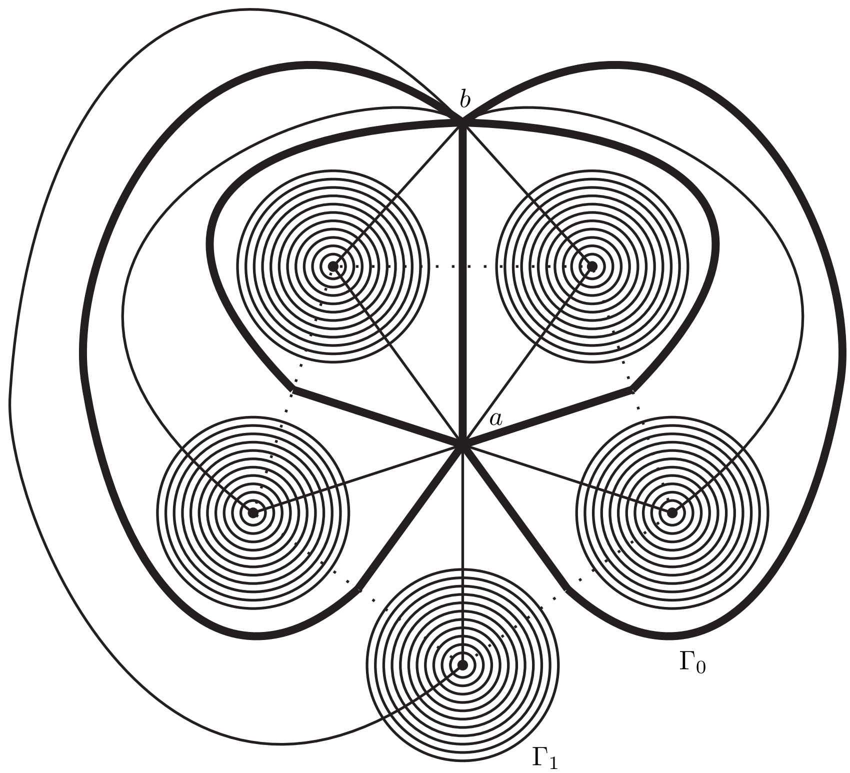
Consider an $\mathrm{IMG}(f)$-orbit in ∂T_t, and let L be the corresponding path connected component of $\mathcal{L}$. The projection is a continuous map $\phi : L \longrightarrow \widehat{\mathbb{C}}$ such that the post-critical set P_f is equal to the union of $\widehat{\mathbb{C}} \setminus \phi(L)$ with the set of critical values of f. The surface L supports a holomorphic structure obtained by pulling back by ϕ the holomorphic structure of $\widehat{\mathbb{C}}$. Accordingly, we can talk about the holomorphic type (hyperbolic or parabolic) of L.

Let us describe, following [Mer08], the *Speiser graph* associated with the map $\phi : L \longrightarrow \widehat{\mathbb{C}}$. Note that P_f is the union of the set $\widehat{\mathbb{C}} \setminus \phi(L)$ and the set of critical values of f. Find a simple closed curve C passing through each point of P_f. It will separate the sphere $\widehat{\mathbb{C}}$ into two parts A and B, while P_f will separate the curve C into $|P_f|$ arcs $C_1, C_2, \ldots, C_{|P_f|}$. Choose points $a \in A, b \in B$, and for every arc C_i connect a to b by a simple curve intersecting C in a single point contained in the interior of C_i. Let Γ_0 be the obtained graph. It subdivides $\widehat{\mathbb{C}}$ into $|P_f|$ regions each of which contains exactly one point from P_f. Add now for every point $z \in P_f$ an infinite sequence of concentric circles around z converging to z and contained in the corresponding region of $\widehat{\mathbb{C}} \setminus \Gamma_0$. After that connect z to a and to b by segments intersecting each of the circles once. See Figure 7. Let Γ_1 be the obtained graph.

The *extended Speiser graph* of $\phi : L \longrightarrow \widehat{\mathbb{C}}$ is the graph $\phi^{-1}(\Gamma_1)$. It is shown in [Mer08], using the results of [Doy84], that L is euclidean if and only if the extended Speiser graph is recurrent. Note that Γ_0 is the union of a set of loops based at a and generating $\pi_1(\widehat{\mathbb{C}} \setminus P_f, a)$. Therefore by Corollary 4.4, the extended Speiser graph contains the orbital graph of $\mathrm{IMG}(f)$. Since subgraphs of recurrent graphs are recurrent, the recurrence of the extended Speiser graph will imply the recurrence of the orbital graph. But it is shown in [LM97, Proposition 4.5] that all path-connected components of $\mathcal{L}$ are parabolic. □

Combining Theorems 7.1 and 6.5 we get the following.

Corollary 7.2. *Suppose that f is a post-critically finite rational function such that a standard action of* $\mathrm{IMG}\, f$ *is generated by an automaton of polynomial activity growth. Then* $\mathrm{IMG}\, f$ *is amenable.*



Example 7.3. The iterated monodromy group of the mating of z^2-1 with the rabbit polynomial is generated by the union of the corresponding iterated monodromy groups of polynomials:

$$\begin{aligned} s_0 &= \sigma(s_0^{-1}, s_0 s_1), & s_1 &= (s_0, 1), \\ r_0 &= \sigma((r_0 r_1)^{-1}, r_0 r_1 r_2), & r_1 &= (r_0, 1), \\ r_2 &= (r_1, 1), \end{aligned}$$

and is a subgroup of $\mathcal{P}_1$. See the Julia set of the mating on Figure 4, which can be realized as the function $f(z) = \frac{(1+i\sqrt{3})/2+z^2}{1-z^2}$. One can not cut it into two disjoint pieces by removing finitely many points, so no standard self-similar action of the iterated monodromy group is generated by bounded automata, see Proposition 3.6.

Example 7.4. An interesting family of rational functions with iterated monodromy groups generated by automata of polynomial activity growth of arbitrary degree is $\{f_c(z) = 1 + \frac{c}{z^2} \ : \ c \in \mathbb{C}\}$.

For example, the iterated monodromy group of $1 + \frac{c}{z^2}$ for $c = \frac{-3-\sqrt{5}}{2} \approx -2.618$ is generated by

$$a_1 = \sigma(a_3, a_2^{-1}), \quad a_2 = (a_1, 1), \quad a_3 = \sigma(1, a_3^{-1}).$$

It is easy to see that it is a subgroup of $\mathcal{P}_1(\mathsf{X})$, see Figure 5.

The Julia set of this rational function is shown on Figure 6.

Example 7.5. The rational function $1-\frac{3+\sqrt{5}}{2z^2}$ has a critical cycle of length 4 (which can be seen as the result of doubling of the critical cycle of the rational function

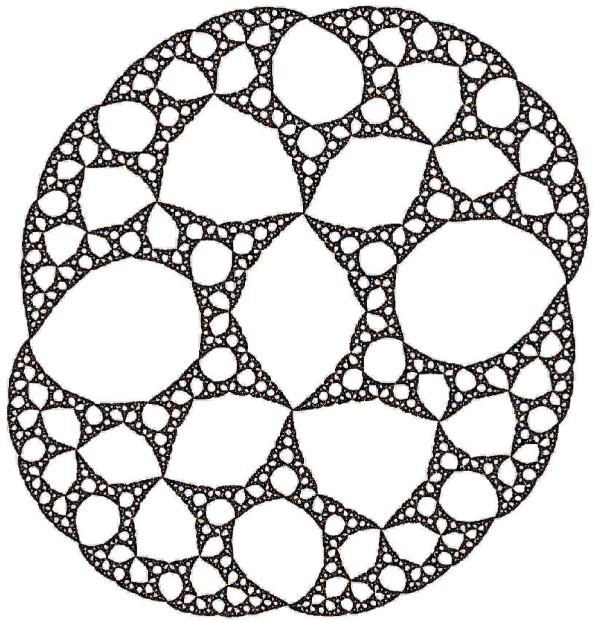


FIGURE 4. Rabbit-Basilica mating

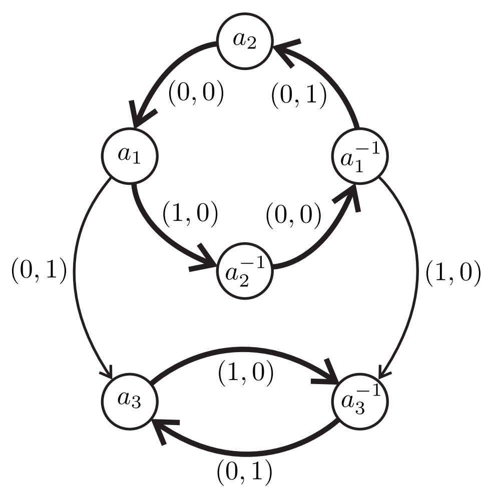


FIGURE 5. Automaton generating IMG $\left(1-\frac{3+\sqrt{5}}{2z^2}\right)$

z^{-2}). Doubling the length of the cycle, i.e., by tuning the rational function by z^2-1, we will get a sequence of rational functions in the family $1+\frac{c}{z^2}$ with iterated monodromy groups in $\mathcal{P}_d$ for arbitrary d. See, for example, the Julia sets of rational functions with iterated monodromy groups generated by automata of quadratic and cubic activity growths on Figure 7.

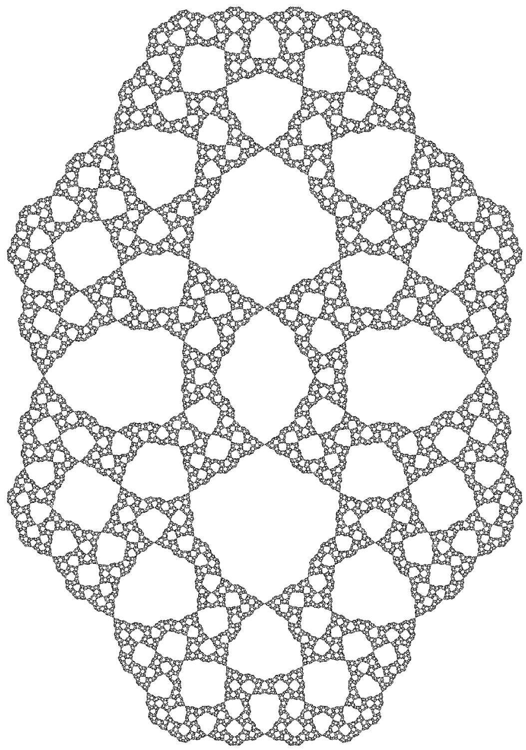


FIGURE 6. The Julia set of $1 - \frac{3+\sqrt{5}}{2z^2}$

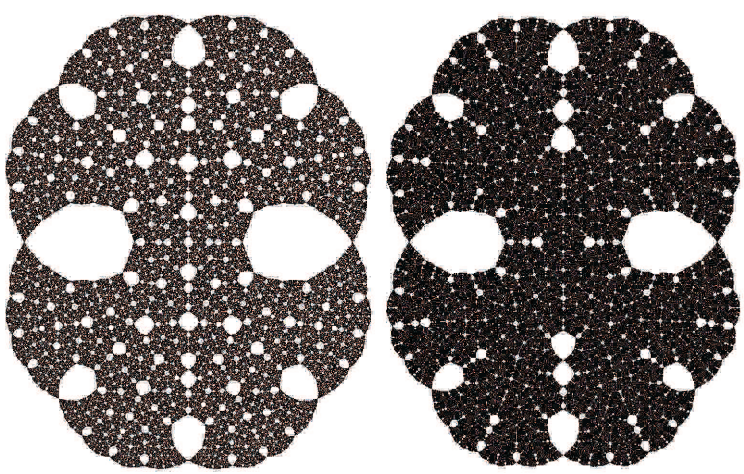


FIGURE 7. Julia sets with iterated monodromy groups contained in $\mathcal{P}_2$ and $\mathcal{P}_3$

Proposition 7.6. *Let f be a post-critically finite rational function such that there exists a tree T on the Riemann sphere such that it contains all post-critical points of f, satisfies $f^{-1}(T) \supset T$, and intersects the Julia set of f in a countable set. Choose a point $t \notin T$, and let ℓ_x be connecting paths as in Proposition 4.2 such that ℓ_x do not intersect T. Then the corresponding action of* $\mathrm{IMG}(f)$ *is a subgroup of* $\mathcal{P}_n(\mathsf{X})$ *for some n.*

For instance, for the rational functions from Examples 7.4 and 7.5 one can take T to be the interval $[a, \infty]$, where a is the smallest post-critical point of the corresponding function.

Proof. The homeomorphism from the limit space of $\mathrm{IMG}(f)$ to the Julia set of f maps the point represented by a sequence $\ldots x_2 x_1$ to the endpoint of the path $\ell_{\ldots x_2 x_1}$ equal to the concatenation of ℓ_{x_1}, a lift of ℓ_{x_2} by f, a lift of ℓ_{x_3} by f^2, etc., see...

The corresponding point belongs to the image of the boundary of the tile $\mathcal{T}$ (see Proposition 3.6, if and only if there exists a sequence $\ldots y_2 y_1 \in \mathsf{X}^{-\omega}$ mapped to the same point of the Julia set and such that $\ell_{\ldots y_2 y_1}^{-1} \ell_{\ldots x_2 x_1}$ is a non-trivial element of the fundamental group of the sphere minus the post-critical set. By the choice of the paths ℓ_x, and by forward invariance of T, the (open) paths $\ell_{\ldots x_2 x_1}$ are disjoint with T. If the loop $\ell_{\ldots y_2 y_1}^{-1} \ell_{\ldots x_2 x_1}$ is disjoint with T, then it is a trivial element of the fundamental group. It follows that the image of the boundary of $\mathcal{T}$ is contained in T. The map from $\mathcal{T}$ to the Julia set is finite-to-one. Consequently, ∂T is countable, and Proposition 3.6 finishes the proof. $\square$

Some properties of such rational functions and their Julia sets... Mention a paper by Schleicher, Hlushchenko, Dudko ???...

8. General case

8.1. **Algebras $\mathcal{A}_p$.** Let $(G, \mathfrak{B})$ be a contracting self-similar group. Recall that we denoted $\mathfrak{B}^* = \bigcup_{n \geq 0} \mathfrak{B}^{\otimes n}$. The set $\mathfrak{B}^*$ is a semigroup with respect to the operation $v \otimes u$, and it contains the group $G = \mathfrak{B}^{\otimes 0}$. The semigroup $\mathfrak{B}^*$ is generated by $\mathsf{X} \cup S$, where S is a generating set of G and X is a basis of $\mathfrak{B}$. It follows from the fact that the right action of G on $\mathfrak{B}$ is free that the semigroup $\mathfrak{B}^*$ is left-cancellative, i.e., that $v \otimes v_1 = v \otimes v_2$ implies $v_1 = v_2$.

Every $v \in \mathfrak{B}^*$ induces therefore an injective map $u \mapsto v \otimes u$ and the associated isometry

$$L_v(\delta_u) = \delta_{v \otimes u}$$

of $\ell^2(\mathfrak{B}^*)$. C^*-algebra related to the operators L_v is one of subjects of the paper [Nek09].

Let X be a basis of $\mathfrak{B}$. Then $\mathfrak{B}^*$ is naturally identified with the direct product $\mathsf{X}^* \times G$. We get a natural topology on $\mathsf{X}^{-\omega} \times G \cup \mathfrak{B}^*$ given by the basis of open sets of the form $\mathsf{X}^{-\omega} \cdot v \cdot g \cup \mathsf{X}^* \cdot v \cdot g$ for $v \in \mathsf{X}^*, g \in G$. This is the natural topology identifying $\mathsf{X}^{-\omega}$ with the boundary of the tree with the set of vertices X^* in which a vertex v is connected to vertices of the form xv for $x \in \mathsf{X}$.

Let $\mathcal{X}_G \cup \mathfrak{B}^*$ be the quotient of the topological space $\mathsf{X}^{-\omega} \times G \cup \mathfrak{B}^*$ by the asymptotic equivalence relation on $\mathsf{X}^{-\omega} \times G$. Consider the algebra $C_0(\mathcal{X}_G \cup \mathfrak{B}^*)$ of compactly supported continuous functions $\mathcal{X}_G \cup \mathfrak{B}^* \longrightarrow \mathbb{C}$. We have a natural inclusion $C_0(\mathcal{X}_G \cup \mathfrak{B}^*) \subset C_0(\mathsf{X}^{-\omega} \times G)$. Note that since $\mathfrak{B}^*$ is dense in $\mathsf{X}^{-\omega} \times G \cup \mathfrak{B}^*$, we have a natural faithful representation ρ of $C_0(\mathsf{X}^{-\omega} \times G \cup \mathfrak{B}^*)$ on $\ell^2(\mathfrak{B}^*)$.

Let $p > 1$. Denote by $\mathcal{A}_p$ the subset of $C_0(\mathcal{X}_G)$ consisting of functions f for which there exists $\tilde{f} \in C_0(\mathcal{X}_G \cup \mathfrak{B}^*)$ such that f is equal to the restriction of $\tilde{f}$ to $\mathcal{X}_G$ and

$$\sum_{v \in \mathfrak{B}^*} |\tilde{f}(u \otimes v) - \tilde{f}(v)|^p < \infty,$$

for every $u \in \mathfrak{B}^*$.

Equivalently, it is the subset of functions f for which there exists an extension $\tilde{f} \in C_0(\mathcal{X}_G \cup \mathfrak{B}^*)$ such that $[\rho(\tilde{f}), L_u]$ belongs to the Schatten ideal $B_p = \{A \in B(\ell^2(\mathfrak{B}^*)) : Tr(|A|^p) < \infty\}$ for every $u \in \mathfrak{B}^*$. Using the fact that B_p is an ideal, it is easy to check that $\mathcal{A}_p$ is a $*$-subalgebra of $C_0(\mathcal{X}_G)$, and one has to check the p-summability condition only for $u \in S \cup \mathsf{X}$, where S is a generating set of G.

Since $\mathcal{A}_p$ is a $*$-sub-algebra, it separates the points of $\mathcal{X}_G$ if and only if it is dense in $C_0(\mathcal{X}_G)$. We obviously have $\mathcal{A}_p \subseteq \mathcal{A}_q$ for $p \leq q$.

It follows that the set of values p for which $\mathcal{A}_p$ is dense is a half-interval (p_c, ∞) or $[p_c, \infty)$ for some p_c. It follows from [BK15, Theorem 3.8] and [Nek03] (see also [Nek05, Theorem 3.8.8]) that p_c is the Ahlfors-regular conformal dimension of $\mathcal{J}_G$ with respect to the natural representation of $\mathcal{J}_G$ as the boundary of a Gromov hyperbolic space $\mathfrak{B}^*/G$, but we will not use this in our paper.

Definition 8.1. Let Γ be a connected graph of bounded degree. Let V and E be the sets of vertices and edges of Γ, respectively. For a function $f : V \longrightarrow \mathbb{R}$, denote by $\nabla f : E \longrightarrow \mathbb{R}$ the function given by $\nabla f(e) = |f(v_1) - f(v_2)|$, where v_1 and v_2 are the endpoints of e. We say that it is *p-parabolic* if for every vertex v of Γ there exists a sequence of finitely -supported functions $f_n : V \longrightarrow \mathbb{R}$ such that $f_n(v) = 1$ and $\sum_{e \in E} \nabla f_n(e)^p \to 0$ as $n \to \infty$.

It is known that recurrence of a simple random walk on a graph is equivalent to its 2-parabolicity, see [Woe00, Theorem 2.12].

Theorem 8.2. *If the algebra $\mathcal{A}_p$ is dense in $C_0(\mathcal{X}_G)$, then the orbital graphs of the action of G on X^ω are p-parabolic.*

Proof. We will prove that the graphs of germs of the action are p-parabolic. The statement of the theorem will follow, since the graphs of germs are finite-to-one covers of the graphs of the action.

Denote by $\mathcal{T}$ the image of $\mathsf{X}^{-\omega} \cup \mathsf{X}^*$ in $\mathcal{X}_G \cup \mathfrak{B}^*$. We know that $\mathcal{T} \cdot \mathcal{N}$ is a neighborhood of $\mathcal{T}$. Since $\mathcal{A}_p$ is dense, there exist a function $f \in C_0(\mathcal{X}_G \cup \mathfrak{B}^*)$ such that $[\rho(f), \pi(g)] \in B_p$ for every $g \in G$, and such that its values on the complement of $\mathcal{T} \cdot \mathcal{N}$ belong to $(-1/3, 1/3)$, and its values on $\mathcal{T}$ belong to $(2/3, 4/3)$. Note that composition $\phi \circ f$ with a Lipschitz function $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ will also satisfy the same p-summability condition of the commutator. In particular, we can compose f with a function ϕ such that $\phi((-1/3, 1/3)) = \{0\}$ and $\phi((2/3, 4/3)) = \{1\}$. Consequently, we may assume that $f \in \mathcal{A}_p$ is supported in $\mathcal{T} \cdot \mathcal{N}$ and is equal to 1 on $\mathcal{T}$.

Let $w = x_1 x_2 \ldots \in \mathsf{X}^\omega$ and consider the graph of germs Γ_w. Let P be the natural quotient map from G to Γ_w given by $g \mapsto (g, w)$. Fix $n \geq 1$, and define $f_n(v) = \sum_{g \in G, (g,w) = v} f(g \cdot x_1 x_2 \ldots x_n)$, where v is a vertex of Γ_w, i.e., a germ at w.

Note that $f_n(g, w) \neq 0$ only for germs of transformations of the form $x_1x_2\ldots x_nu \mapsto y_1y_2\ldots y_nh(u)$ for $y_1y_2\ldots y_n \in \mathsf{X}^*$ and $h \in \mathcal{N}$. In particular, f_n is finitely supported. Let $g \cdot x_1x_2\ldots x_n \in \mathcal{T}\cdot\mathcal{N}$. Then the germ (g, w) is uniquely determined by the element $g \cdot x_1x_2\ldots x_n \in \mathfrak{B}^{\otimes n}$ and the element $g|_{x_1x_2\ldots x_n} \in \mathcal{N}$. Hence two such elements g_1, g_2 define the same germ (g_i, w) but different elements $g_i \cdot x_1x_2\ldots x_n \in \mathcal{T}\cdot\mathcal{N}$ only if $g_i|_{x_1x_2\ldots x_n} \in \mathcal{N}$ are different. It follows that the sum in the definition of f_n has at most $|\mathcal{N}|$ non-zero summands.

We have then for every edge $(\gamma, s\cdot\gamma)$ of the graph of germs

$$|f_n(\gamma) - f_n(s\cdot\gamma)|^p = \left|\sum_{g\in G,(g,w)=\gamma} f(g\cdot x_1x_2\ldots x_n) - f(sg\cdot x_1x_2\ldots x_n)\right|^p \leq$$

$$|\mathcal{N}|^{2(p-1)} \sum_{g\in G,(g,w)=\gamma} |f(g\cdot x_1x_2\ldots x_n) - f(sg\cdot x_1x_2\ldots x_n)|^p.$$

Consequently, $\sum |\nabla f_n|^p$ is bounded from above by

$$|\mathcal{N}|^{2(p-1)} \sum_{g\in G, s\in S} |f(g\cdot x_1x_2\ldots x_n) - f(sg\cdot x_1x_2\ldots x_n)|^p.$$

But the latter approaches zero as $n \to \infty$, since ∇f is p-summable. □

Proposition 8.3. *If there exists a p-contracting model of G, then $\mathcal{A}_p$ is dense.*

Proof. Let Γ be a connected metric graph with a right proper co-compact isometric G-action, and let $I : \Gamma\otimes\mathfrak{B} \longrightarrow \Gamma$ be a p-contracting G-equivariant map. It induces continuous maps $I_n^\infty : \mathcal{X}_G \longrightarrow \Gamma\otimes\mathfrak{B}^{\otimes n}$ for every n. Let $\mathcal{L}_n$ be the set of compositions of I_n^∞ with Lipschitz compactly supported functions $\Gamma\otimes\mathfrak{B}^{\otimes n} \longrightarrow \mathbb{R}$. Since the map I is Lipschitz, we have $\mathcal{L}_{n+1} \supset \mathcal{L}_n$. Let $\mathcal{L} = \bigcup_{n\geq 0}\mathcal{L}_n$. The set $\mathcal{L}$ separates the points of $\mathcal{X}_G$, since the space $\mathcal{X}_G$ is the inverse limit of the graphs $\Gamma\otimes\mathfrak{B}^{\otimes n}$ with respect to the maps I_n^m. It is enough therefore to show that $\mathcal{L}_n \subset \mathcal{A}_p$ for every n.

Let $f : \Gamma\otimes\mathfrak{B}^{\otimes n} \longrightarrow \mathbb{R}$ be a compactly supported Lipschitz function. Pick a sequence $w_0 = \ldots x_2x_1 \in \mathsf{X}^{-\omega}$, and define a map $\tilde{f} : \mathfrak{B}^* \longrightarrow \mathbb{R}$ by $\tilde{f}(v) = f(I_n^\infty(w_0\otimes v))$.

The map $I_n^\infty : \mathcal{X}_G \longrightarrow \Gamma\otimes\mathfrak{B}^{\otimes n}$ is proper (as an equivariant map between proper co-compact G-spaces). It follows that the support of $\tilde{f}(w)$ intersected with $\mathcal{X}_G$ is compact. In particular, the set F of elements of g such that $\tilde{f}(w\cdot g) \neq 0$ for some $x_i \in \mathsf{X}$ is finite.

A point $v \in \mathfrak{B}^{\otimes k}$ belongs to the support of $\tilde{f}$ if and only if $w_0\otimes v \in \mathcal{X}_G$ belongs to the support of $\tilde{f}$. This implies that $v = x_k\ldots x_2x_1\cdot g$ for some $x_i \in \mathsf{X}$ and $g \in F$. Consequently, the support of $\tilde{f}$ is compact. The function $\tilde{f}|_{\mathcal{X}_G}$ belongs to $\mathcal{L}_n$, by definition. The function $\tilde{f}$ is its compactly supported extension to $\mathcal{X}_G \cup \mathfrak{B}^*$.

Let us show that for every $g \in G$ and $x \in \mathsf{X}$ the functions $\tilde{f}(v) - \tilde{f}(g\cdot v)$ and $\tilde{f}(v) - \tilde{f}(x\otimes v)$ are p-summable. Let L be the Lipschitz constant for f and let λ be the p-contraction coefficient for I. Then we have for every k such that $g|_v \in \mathcal{N}$

for all $v \in \mathsf{X}^k$, we have

$$\sum_{v\in\mathfrak{B}^{\otimes k}} |\tilde{f}(v) - \tilde{f}(g\cdot v)|^p = \sum_{v\in\mathfrak{B}^{\otimes k}} |f(I_n^\infty(w_0\otimes v)) - f(I_n^\infty(w_0\cdot g\otimes v))|^p =$$
$$\sum_{v\in\mathsf{X}^k, h\in G} |f(I_n^\infty(w_0\otimes v\cdot h)) - f(I_n^\infty(w_0\cdot g\otimes v\cdot h))|^p =$$
$$\sum_{v\in\mathsf{X}^k, h\in F\cup\mathcal{N}F} |f(I_n^\infty(w_0\otimes v\cdot h)) - f(I_n^\infty(w_0\cdot g\otimes v\cdot h))|^p \le$$
$$\sum_{v\in\mathsf{X}^k, h\in F\cup\mathcal{N}F} L^p|I_n^\infty(w_0\otimes v\cdot h) - I_n^\infty(w_0\cdot g\otimes v\cdot h)|^p =$$
$$|F\cup\mathcal{N}F|\sum_{v\in\mathsf{X}^k} L^p|I_n^\infty(w_0\otimes v) - I_n^\infty(w_0\cdot g\otimes v)|^p \le C\lambda^k$$

for some constant C not depending on k, by Proposition 5.3. Similarly, if we denote $w_1 = w_0\otimes x$ for $x\in\mathsf{X}$, then we have for all $k\ge 0$:

$$\sum_{v\in\mathfrak{B}^{\otimes k}} |\tilde{f}(v) - \tilde{f}(x\otimes v)|^p = \sum_{v\in\mathfrak{B}^{\otimes k}} |f(I_n^\infty(w_0\otimes v)) - f(I_n^\infty(w_1\otimes v))|^p =$$
$$\sum_{v\in\mathsf{X}^k, h\in G} |f(I_n^\infty(w_0\otimes v\cdot h)) - f(I_n^\infty(w_1\otimes v\cdot h))|^p =$$
$$\sum_{v\in\mathsf{X}^k, h\in F\cup\mathcal{N}F} |f(I_n^\infty(w_0\otimes v\cdot h)) - f(I_n^\infty(w_1\otimes v\cdot h))|^p \le$$
$$\sum_{v\in\mathsf{X}^k, h\in F\cup\mathcal{N}F} L^p|I_n^\infty(w_0\otimes v\cdot h) - I_n^\infty(w_1\otimes v\cdot h)|^p =$$
$$|F\cup\mathcal{N}F|\sum_{v\in\mathsf{X}^k} L^p|I_n^\infty(w_0\otimes v) - I_n^\infty(w_1\otimes v)|^p \le C\lambda^k$$

for some C not depending on k. It follows that $\tilde{f}$ belongs to $\mathcal{A}_p$. □

8.2. **p-contracting models for subgroups of $\mathcal{P}_n(\mathsf{X})$.**

Theorem 8.4. *Every contracting self-similar subgroup of $\mathcal{P}_d(\mathsf{X})$ is amenable.*

The theorem follows directly from Proposition 8.3, Theorems 8.2 and 6.5, and the following theorem.

Theorem 8.5. *If G is a self-similar finitely generated contracting subgroup of $\mathcal{P}_n(\mathsf{X})$, then it has a p-contracting model for every $p > 1$.*

Proof. We prove it by induction on n. Suppose that it is true for all values smaller than n. Let $G \le \mathcal{P}_n(\mathsf{X})$ be a self-similar contracting group. Let $\mathfrak{B}$ be the associated G-biset.

Suppose at first that G is generated by its nucleus $\mathcal{N}$. We pass to the power of the alphabet such that every non-trivial cycle of $\mathcal{N}$ is a loop.

Let $\mathcal{N}_0 = \mathcal{N}\cap\mathcal{P}_{n-1}(\mathsf{X})$, and let $G_0 < G$ be the group generated by $\mathcal{N}_0$. (Note that the nucleus of G_0 is in general smaller than $\mathcal{N}_0$, and is obtained from $\mathcal{N}_0$ by removing vertices that are not reachable from cycles of $\mathcal{N}_0$.) Let $\mathfrak{B}_0 \subset \mathfrak{B}$ be the corresponding G_0-biset. Let $\mathcal{X}$ and $\mathcal{X}_0$ be the limit G-space and limit G_0-space, respectively.

Let $p > 1$ be arbitrary. By the inductive assumption, after passing to an iteration $\mathfrak{B}_0^{\otimes n}$, there exists a connected graph Γ_0 with a proper right G_0-action, and a map $I : \Gamma_0 \otimes \mathfrak{B}_0 \longrightarrow \Gamma_0$, such that

$$\operatorname{ess\,sup}_{t\in\Gamma_0} \sum_{x\in\mathsf{X}} \left|\frac{dI(t\otimes x)}{dt}\right|^p = \lambda < 1.$$

Let $I_0^\infty : \mathcal{X}_0 \longrightarrow \Gamma_0$ be the corresponding projection map. For every letter $x \in \mathsf{X}$ we have the corresponding point $I^\infty(x^{-\omega})$. (We denote $x^{-\omega} = \ldots xxx$.) After replacing Γ_0 by $\Gamma_0 \otimes \mathfrak{B}_0^{\otimes k}$, if necessary, we may assume that $I^\infty(x_1^{-\omega}) \neq I^\infty(x_2^{-\omega})$ when the points of $\mathcal{X}_0$ represented by $x_i^{-\omega}$ are different. We also assume that the points $I^\infty(x^{-\omega})$ are vertices of Γ_0 and that combinatorial distance between any two of them is at least 2 (this can be achieved just by subdividing the edges of Γ_0 in a G_0-equivariant way).

Consider now $\Gamma_0 \otimes_{G_0} G$, where G is seen as a left G_0-set. The connected components of $\Gamma_0 \otimes_{G_0} G$ correspond to the right G-cosets of G_0. Namely, two points $v_1 \otimes_{G_0} g_1$ and $v_2 \otimes_{G_0} g_2$ belong to the same connected component of $\Gamma_0 \otimes_{G_0} G$ if and only if $G_0 g_1 = G_0 g_2$.

Add a new vertex v_ξ for every point $\xi \in \mathcal{X}$ that can be represented in the form $x^{-\omega} \cdot g$ for $x \in \mathsf{X}$, and connect it by an edge of length L (to be chosen later) to the point $I^\infty(x^{-\omega}) \otimes_{G_0} g$ of $\Gamma_0 \otimes_{G_0} G$. Let Γ_1 be the obtained graph. The group G acts naturally on Γ_1. Note that if v_ξ is connected to a vertex $I^\infty(x^{-\omega}) \otimes_{G_0} g$, then ξ can be represented as $x^{-\omega} \cdot g$. This implies that Γ_1 has bounded degree, since the sizes of the asymptotic equivalence classes in $\mathsf{X}^{-\omega} \times G$ are bounded.

Let $g \in \mathcal{N} \setminus \mathcal{N}_0$. Then there exists a unique pair of letters $x, y \in \mathsf{X}$ such that $g \cdot x = y \cdot g$. Then for every $h \in G$ the sequences $x^{-\omega} \cdot h$ and $y^{-\omega} \cdot gh$ represent the same point ξ of $\mathcal{X}$. The points $I_0^\infty(x^{-\omega} \cdot h)$ and $I_0^\infty(y^{-\omega} \cdot gh)$ belong to the connected components of $\Gamma_0 \otimes_{G_0} G$ corresponding to the cosets $G_0 h$ and $G_0 gh$, and are both connected by the edges of Γ_1 to the vertex v_ξ. Since the set $G_0 \cup (\mathcal{N} \setminus \mathcal{N}_0)$ generates G, it follows that Γ_1 is connected.

Let us define a self-similarity $J : \Gamma_1 \otimes_G \mathfrak{B} \longrightarrow \Gamma_1$. We set its restriction to $\Gamma_0 \otimes_{G_0} G$ to coincide with the previously constructed self-similarity (induced by $I \otimes_{G_0} Id$). Let $y \in \mathsf{X}$. Consider a vertex vertex v_ξ of Γ_1 not belonging to $\Gamma_0 \otimes_{G_0} G$ and a letter $y \in \mathsf{X}$. Let us define $J(\cdot \otimes y)$ on the union of the edges adjacent to v_ξ. Each such an edge connects v_ξ with a point $I_0^\infty(x^{-\omega}) \otimes_{G_0} g$, where $x^{-\omega} \cdot g$ represents ξ. We have to define the action of $J(\cdot, y)$ on the edge connecting v_ξ with $I_0^\infty(x^{-\omega}) \otimes_{G_0} g$.

We consider two cases:

Case I. Suppose that $g(y) \neq x$. Then $\xi \otimes y$ is represented by $x^{-\omega} g(y) \cdot g|_x$. Let us show that $x^{-\omega} g(y) \cdot g|_x$ can not be represented by $z^{-\omega} \cdot h$ for any $z \in \mathsf{X}$ and $h \in G$. Suppose that it is. Then the asymptotic equivalence between $x^{-\omega} g(y) \cdot g|_x$ and $z^{-\omega} \cdot h$ is given by a loop passing through $f \in \mathcal{N}$ such that $f \cdot x = z \cdot f$, but then we will have $f \cdot g(y) g|_x = z \cdot h$, which contradicts the fact that $f^{-1}(z) = x$. Consequently, the point $\xi \otimes y \in \mathcal{X}$ is not one of the points used to define vertices of Γ_1 not contained in $\Gamma_0 \otimes_{G_0} G$.

Consider the set of vertices of $\Gamma_0 \otimes_{G_0} G$ adjacent to v_ξ. Let $I^\infty(x_i^{-\omega}) \otimes_{G_0} g_i$ for $i = 1, 2$ be two of them. Then we have $\xi = x_1^{-\omega} \cdot g_1 = x_2^{-\omega} \cdot g_2$ in $\mathcal{X}$, but $x_i^{-\omega} \otimes_{G_0} g_i$ are different in $\mathcal{X}_0 \otimes_{G_0} G$. There exists $h \in \mathcal{N} \setminus \mathcal{N}_0$ such that $h \cdot x_1 = x_2 \cdot h$ and

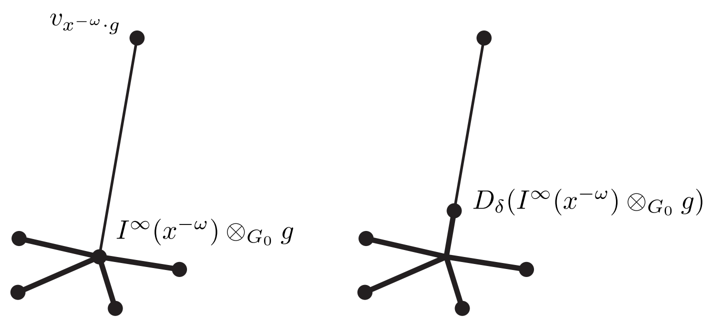


Figure 8. The map D_δ

$hg_1 = g_2$. Consider the corresponding sequences representing $\xi\otimes y$ in $\mathcal{X}_0\otimes_{G_0} G$:

$$(x_1^{-\omega}\otimes_{G_0} g_1)\otimes y = (x_1^{-\omega}g_1(y))\otimes_{G_0} g_1|_y$$

and

$$(x_2^{-\omega}\otimes_{G_0} hg_1)\otimes y = (x_2^{-\omega}(hg_1)(y))\otimes_{G_0} h|_{g_1(y)}g_1|_y.$$

Since $h|_{g_1(y)}\in G_0$, these two points will belong to the same connected component of $\Gamma_0\otimes_{G_0} G$, and there is a uniform bound B on the distance between such points (e.g., the diameter of $I^{\infty}(\mathsf{X}^{-\omega}\cdot\mathcal{N}_0)$ in Γ_0). We can define then the image of $v_\xi\otimes y$ to be any of these points, and map the corresponding edges linearly to the geodesics. Then the length of the images of the edges adjacent to v_ξ will be bounded above by B. (Note that B does not depend on L.)

Case II. Suppose that $g(y) = x$. Then we have $\xi\otimes y = x^{-\omega}\cdot g|_y$, so we can map the vertex v_ξ to the vertex $v_{\xi\otimes y}$. Suppose that $I^{\infty}(x_1^{-\omega}\otimes_{G_0} g_1)$ is a vertex adjacent to v_ξ. Then $x_1^{-\omega}\cdot g_1$ and $x^{-\omega}\cdot g$ both represent ξ, so that there exists $h\in\mathcal{N}$ such that $h\cdot x_1 = x\cdot h$ and $hg_1 = g$. Then we have $I(I^{\infty}(x_1^{-\omega}\otimes_{G_0} g_1)\otimes y) = I^{\infty}(x_1^{-\omega}g_1(y)\otimes_{G_0} g_1|_y)$. We have $g_1(y) = h^{-1}g(y) = h^{-1}(x) = x_1$, so that $I(I^{\infty}(x_1^{-\omega}\otimes_{G_0} g_1\otimes y)$ is a vertex of $\Gamma_0\otimes_{G_0} G$ connected to $v_{\xi\otimes y}$.

It follows that the edges adjacent to v_ξ can be mapped by J accordingly without any change of their length.

We have constructed a G-equivariant map $J:\Gamma_1\otimes\mathfrak{B}\longrightarrow\Gamma_1$. We will now modify it to a new equivariant map J' by post-composing it with the following map $D_\delta:\Gamma_1\longrightarrow\Gamma_1$. The map D_δ will be identical on all vertices of Γ_1 except for the vertices of the form $I^{\infty}(x^{-\omega})\otimes_{G_0} g$. We define $D_\delta(I^{\infty}(x^{-\omega})\otimes_{G_0} g)$ to be the point on the edge connecting $I^{\infty}(x^{-\omega})\otimes_{G_0} g$ to $v_{x^{-\omega}\cdot g}$ on distance δ from $I^{\infty}(x^{-\omega})\otimes_{G_0} g$. We re-scale linearly every edge adjacent to $I^{\infty}(x^{-\omega})\otimes_{G_0} g$ accordingly, see Figure 8. It is easy to see that D_δ is G-equivariant. Denote $\tilde{J} = D_\delta\circ J$.

If e is an edge of $\Gamma_0\otimes_{G_0} G$ of length l, then it will be mapped by D_δ either to itself, or to an edge of length at most $l+\delta$. It follows that if l_0 be the minimum of the lengths of the edges of Γ_0, then $\left|\frac{dD_\delta(t)}{dt}\right|\le\frac{l_0+\delta}{l_0}$ for all t (where the derivative exists).

Let e be an edge of Γ_1. If $e\in\Gamma_0\otimes_{G_0} G$, then for almost every $t\in e$ we have

$$\sum_{x\in\mathsf{X}}\left|\frac{d\tilde{J}(t\otimes x)}{dt}\right|^p\le\left(\frac{l_0+\delta}{l_0}\right)^p\sum_{x\in\mathsf{X}}\left|\frac{dI(t\otimes x)}{dt}\right|^p\le\left(\frac{l_0+\delta}{l_0}\right)^p\lambda.$$

Since $\left(\frac{l_0+\delta}{l_0}\right)^p \to 1$ as $\delta \to 0$, we can choose δ such that $\left(\frac{l_0+\delta}{l_0}\right)^p \lambda < 1$. Let us fix this δ.

If e does not belong to $\Gamma_0 \otimes_{G_0} G$, then for almost every $t \in e$ we have

$$\sum_{x \in \mathsf{X}} \left|\frac{d\tilde{J}(t \otimes x)}{dt}\right|^p \le (|\mathsf{X}| - 1)\left(\frac{l_0+\delta}{l_0}\right)^p \frac{B^p}{L^p} + \left(\frac{L-\delta}{L}\right)^p = \frac{C}{L^p} + \left(1 - \frac{\delta}{L}\right)^p,$$

where C is a constant not depending on L. The function $f(t) = Ct^p + (1-\delta t)^p$ is equal to 1 at $t = 0$. Its derivative $\frac{df}{dt}(t) = pCt^{1-p} - \delta p(1-\delta t)^{p-1}$ is $-p\delta < 0$ at $t = 0$. It follows that there exists $t > 0$ such that $f(t) < 1$. Taking $L = t^{-1}$, we find $L > 0$ such that $\frac{C}{L^p} + \left(1 - \frac{\delta}{L}\right)^p < 1$. We see that $\tilde{J}$ is p-contracting for the given choices of δ and L.

Suppose now that G is not generated by its nucleus. Let G_0 be the subgroup generated by the nucleus $\mathcal{N}$. Let $\mathcal{A}$ be the set of states of the automaton generating G which do not belong to G_0. After passing to a power of alphabet, we may assume that for every $g \in \mathcal{A}$ and every $x \in \mathsf{X}$ we have $g|_x \in \mathcal{N}$. In particular, the image of G under the wreath recursion belongs to $G_0^{\mathsf{X}} \rtimes S_{\mathsf{X}}$. By the proven above (and by the inductive hypothesis), there exists a metric graph Γ_0 with a proper and co-compact isometric right G_0-action and a p-contracting G_0-equivariant map $I : \Gamma_0 \otimes \mathfrak{B}_0 \longrightarrow \Gamma_0$. Let λ be the p-contraction coefficient for I.

Consider the graph $\Gamma_0 \otimes_{G_0} G$. Choose an arbitrary vertex $w \in \Gamma_0$, and connect for every $h \in \mathcal{A}$ and $g \in G$ the vertex $w \otimes_{G_0} g$ to $w \otimes_{G_0} hg$ by an edge of length M that will be chosen later. Let Γ_1 be the obtained graph. It is connected, since $\mathcal{A} \cup G_0$ generates G. Let us define $J : \Gamma_1 \otimes \mathfrak{B} \longrightarrow \Gamma_1$. Since the set of vertices of the graphs Γ_1 and $\Gamma_0 \otimes_{G_0} G$ are the same, we define J on the set of vertices to be equal to $I \otimes Id$. Consider a new edge e connecting $w \otimes_{G_0} g$ to $w \otimes_{G_0} hg$ for $h \in \mathcal{A}$ and $g \in G$, and let $x \in \mathsf{X}$. Then the image $J(e \otimes x)$ of the edge e has to connect the points $I(w \otimes g(x)) \cdot g|_x$ and $I(w \otimes hg(x)) \cdot h|_{g(x)} g|_x$. Note that $h|_{g(x)} \in \mathcal{N}$, so that the points $I(w \otimes g(x)) \cdot g|_x$ and $I(w \otimes hg(x)) \cdot h|_{g(x)} g|_x$ belong to the same connected component of $\Gamma_0 \otimes_{G_0} G$. We can map therefore the edge e by $I(\cdot \otimes x)$ linearly to the geodesic connecting them. This will define our map $J : \Gamma_1 \otimes \mathfrak{B} \longrightarrow \Gamma_1$.

The map J coincides with I on the connected components of $\Gamma_0 \otimes_{G_0} G$, which are invariant under $I(\cdot \otimes x)$. This proves that J is p-contracting on the edges of $\Gamma_0 \otimes_{G_0} G$. The set $\{I(w \otimes y) \cdot f) \ : \ y \in \mathsf{X}, f \in \mathcal{N}\}$ is finite, hence there is a uniform upper bound B on the length of the $I(e \otimes x)$ for the edges $e \in \Gamma_1 \setminus \Gamma_0 \otimes_{G_0} G$. Consequently, for any such an edge e we have

$$\sum_{x \in \mathsf{X}} \left|\frac{dJ(t \otimes x)}{dt}\right|^p \le |\mathsf{X}| \cdot \frac{B^p}{L^p}.$$

By choosing sufficiently large L we can achieve p-contraction. □

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