MATH 416, Modern Algebra II

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Simplices, reminder

A *d-dimensional simplex* is given by a set of vertices $v_0, v_1, \dots, v_d$. An *oriented simplex* is an ordered set of vertices $(v_0, v_1, \dots, v_d)$. Two orderings define the same oriented simplex if the permutation transforming one ordering into the other is even. Otherwise their orientation is *opposite*. For example, (v_0, v_1) and (v_1, v_0) are two different orientations of a 1-dim simplex (of a segment). Simplices (v_0, v_1, v_2), (v_1, v_2, v_0), (v_0, v_1, v_2) have the same orientation; $(v_0, v_2, v_1), (v_2, v_1, v_0), (v_1, v_0, v_2)$ have the opposite orientation.

Simplicial complexes, reminder

A *simplicial complex* Δ is a given by a set of vertices V and the set of simplices on V (i.e., finite subsets of V) such that if $S = \{v_0, v_1, \dots, v_d\}$ is a simplex of Δ, then every subset of S is also a simplex of Δ (the subsets are called *faces* or *subsimplices* of S). The *dimension* of Δ is the maximal dimension of its simplices.

For example, a one-dimensional simplicial complex is a graph. (A 0-dimensional simplicial complex is just a set of vertices.)

Chains and their boundaries, reminder

Let Δ be a simplicial complex, where we choose an orientation for each simplex (in an arbitrary way). Its *group of d-dimensional chains* is the free abelian group generated by the d-dimensional simplices of Δ. So, a *chain* is a formal linear combination $m_1S_1 + m_2S_2 + \cdots + m_kS_k$, where $m_i \in \mathbb{Z}$ and S_i are oriented d-simlices of Δ. We denote the group of d-dimensional chains by $C_d(\Delta)$. If S is an oriented simplex, then we interpret $-S$ as the simplex with the opposite orientation.

If $S = (v_0, v_1, v_2, \ldots, v_d)$ is an oriented d-simplex, then its *boundary* is the chain

$$\partial_d S = (v_1, v_2, \ldots, v_d) - (v_0, v_2, \ldots, v_d) + \cdots + (-1)^k(v_0, v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_d) + \cdots + (-1)^d(v_0, v_1, \ldots, v_{d-1}) \in C_{d-1}(S).$$

It agrees with the usual notion of the *oriented boundary* of a simplex.

Homology

We get a sequence of abelian groups and homomorphisms

$$C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} C_2 \xleftarrow{\partial_3} \cdots$$

We have checked in the previous lecture that $\partial_n \circ \partial_{n+1}$ maps everything to zero.
In other words, $\operatorname{im} \partial_{n+1} \leq \ker \partial_n$. The elements of $\operatorname{im} \partial_{n+1} =: B_n(\Delta)$ are called (n-dimensional) *boundaries*. The elements of $\ker \partial_n =: Z_n(\Delta)$ are called (n-dimensional) *cycles*. Every boundary is a cycle, so that $B_n(\Delta) \leq Z_n(\Delta)$.
We assume that $C_n(\Delta) = 0$ when n is greater than the dimension of Δ. In particular, $\partial_n : C_n(\Delta) \longrightarrow C_{n-1}(\Delta)$ is then the zero homomorphism, so we have $B_d(\Delta) = 0$ if d is equal to or larger than the dimension of Δ.
Similarly, we assume that $C_{-1}(\Delta) = 0$, so that $\partial_0 : C_0(\Delta) \longrightarrow C_{-1}(\Delta)$ is also constant zero. Therefore, all elements of $C_0(\Delta)$ are considered to be cycles.

Homology

The nth *homology group* is the factor group

$$H_n(\Delta) = Z_n(\Delta)/B_n(\Delta) = \ker \partial_n / \operatorname{im} \partial_{n+1}.$$

It measures how much is there cycles that are not boundaries. Informally, it measure number of n-dimensional "holes."
For example, $H_0(\Delta)$ is the factor of $C_0(\Delta)$ by the subgroup generated by $A - B$ for all edges AB of Δ. In other words, $H_0(\Delta)$ is obtained from the free abelian group generated by the vertices by identifying any two vertices connected by an edge. We conclude that $H_0(\Delta)$ is the free abelian group generated by the *connected components* of Δ. In particular, Δ is connected if and only if $H_0(\Delta) = \mathbb{Z}$.

Example: A ring of edges

Suppose that Δ is the graph consisting of one cycle of edges $e_1, e_2, \dots, e_n$ of length n (i.e., an n-gon). Then $C_1(\Delta)$ is $\mathbb{Z}^n$ generated by n edges e_i, $C_0(\Delta)$ is $\mathbb{Z}^n$ generated by the vertices of the n-gon.
We know that $H_0(\Delta) = \mathbb{Z}$, since Δ is connected. We have $C_2(\Delta) = 0$, since Δ is one-dimensional. Consequently, $B_1(\Delta) = 0$.

Example: A ring of edges

Let $v_1, v_2, \ldots, v_n$ be the vertices of Δ, so that $e_i = v_i v_{i+1}$ and $e_n = v_n v_1$. A chain $m_1 e_1 + m_2 e_2 + \cdots + m_n e_n$ is a cycle if its boundary $m_1(v_1 - v_2) + m_2(v_2 - v_3) + \cdots + m_n(v_n - v_1)$ is equal to 0, Rearranging, we get that the boundary is $(m_1 - m_n)v_1 + (m_2 - m_1)v_2 + \cdots + (m_n - m_{n-1})v_n$. Since $\{v_1, \ldots, v_n\}$ is a basis of $C_1(\Delta)$, this happens if and only if $m_1 - m_n = m_2 - m_1 = \ldots = m_n - m_{n-1} = 0$, i.e., if and only if $m_1 = m_2 = \ldots = m_n$. We see that $Z_1(\Delta)$ is the set of chains of the form $m(e_1 + e_2 + \cdots + e_n)$, i.e., is the cyclic subgroup generated by $e_1 + e_2 + \cdots + e_n$. Consequently, $H_1(\Delta) = Z_1(\Delta)/0 \cong \mathbb{Z}$. This corresponds to the intuition that Δ has one "hole."

Figure eight

Consider now the graph Δ obtained by taking boundaries of two triangles ABC and AXY with one common vertex A. It has two "holes." Let us compute $H_1(\Delta)$.

A 1-dimensional chain is any expression of the form $m_1AB + m_2BC + m_3CA + k_1AX + k_2XY + k_3YA$. Its boundary is $m_1(A-B)+m_2(B-C)+m_3(C-A)+k_1(A-X)+k_2(X-Y)+k_3(Y-A) = (m_1-m_3+k_1-k_3)A+(m_2-m_1)B+(m_3-m_2)C+(k_2-k_1)X+(k_3-k_2)Y$. It is equal to 0 if and only if

$$\begin{cases} m_1 - m_3 + k_1 - k_3 & = & 0 \\ m_2 - m_1 & = & 0 \\ m_3 - m_2 & = & 0 \\ k_2 - k_1 & = & 0 \\ k_3 - k_2 & = & 0 \end{cases}$$

Figure eight

$$\begin{cases} m_1 - m_3 + k_1 - k_3 & = & 0 \\ m_2 - m_1 & = & 0 \\ m_3 - m_2 & = & 0 \\ k_2 - k_1 & = & 0 \\ k_3 - k_2 & = & 0 \end{cases}$$

We see that the chain is a cycle if and only if $m_1 = m_2 = m_3$ and $k_1 = k_2 = k_3$. It follows that the group of cycles is the set of chains of the form $mAB + mBC + mCA + kAX + kXY + kYA$, i.e., it is the free abelian group generated by $AB + BC + CA$ and $AX + XY + YA$. Since the group of boundaries is 0, it follows that $H_2(\Delta) = \mathbb{Z}^2$.

Sphere

Let Δ be the boundary of a tetrahedron $ABCD$. Then $C_2(\Delta)$ is $\mathbb{Z}^4$ generated by BCD, CAD, ABD, ACB, $C_1(\Delta)$ is $\mathbb{Z}^6$ generated by AB, AC, AD, BC, BD, CD, and $C_0(\Delta)$ is $\mathbb{Z}^4$ generated by A, B, C, D. We have $H_0(\Delta) = \mathbb{Z}$, since Δ is connected. Let us find $H_1(\Delta)$. The group of boundaries $B_1(\Delta)$ is generated by the boundaries of the faces of the tetrahedron, i.e., by
$CD + DB + BC, AD + DC + CA, BD + DA + AB, CB + BA + AC$.
Consequently, it is the group of all chains of the form

$$m_1(CD + DB + BC) + m_2(AD + DC + CA)+ \\ m_3(BD + DA + AB) + m_4(CB + BA + AC) = \\ (m_3 - m_4)AB + (m_4 - m_2)AC + (m_2 - m_3)AD+ \\ (m_1 - m_4)BC + (m_3 - m_1)BD + (m_1 - m_2)CD.$$

Sphere

$$m_1(CD + DB + BC) + m_2(AD + DC + CA) + m_3(BD + DA + AB) + m_4(CB + BA + AC) = (m_3 - m_4)AB + (m_4 - m_2)AC + (m_2 - m_3)AD + (m_1 - m_4)BC + (m_3 - m_1)BD + (m_1 - m_2)CD.$$

Denote $k_1 = m_3 - m_4$, $k_2 = m_4 - m_2$, $k_4 = m_1 - m_4$. Then $m_3 - m_2 = k_1 + k_2$, $m_3 - m_1 = k_1 + k_4$, and $m_1 - m_2 = k_2 + k_4$. We get that the group of boundaries is the set of chains of the form

$$k_1AB + k_2AC + (-k_1 - k_2)AD + k_4BC + (k_1 + k_4)BD + (k_2 + k_4)CD$$

Sphere

A chain $k_1AB + k_2AC + k_3AD + k_4BC + k_5BD + k_6CD$ is a cycle if and only if its boundary is zero. The boundary is

$$k_1A - k_1B + k_2A - k_2C + k_3A - k_3D + k_4B - k_4C + k_5B - k_5D + k_6C - k_6D =$$
$$(k_1 + k_2 + k_3)A + (-k_1 + k_4 + k_5)B + (-k_2 - k_4 + k_6)C + (-k_2 - k_4 + k_6)D,$$

so the condition to be a cycle is

$$\begin{cases} k_1 + k_2 + k_3 &= 0 \\ -k_1 + k_4 + k_5 &= 0 \\ -k_2 - k_4 + k_6 &= 0 \\ -k_2 - k_4 + k_6 &= 0 \end{cases}$$

Sphere

$$\begin{cases} k_1 + k_2 + k_3 &= 0 \\ -k_1 + k_4 + k_5 &= 0 \\ -k_2 - k_4 + k_6 &= 0 \\ -k_2 - k_4 + k_6 &= 0 \end{cases}$$

This is equivalent to the condition $k_3 = -k_1 - k_2$, $k_5 = k_1 - k_2$, $k_6 = k_2 + k_4$, which is the same condition as for belonging to $B_1(\Delta)$. We see that $B_1(\Delta) = Z_1(\Delta)$, hence $H_1(\Delta) = 0$. This corresponds to the fact that there are no one-dimensional holes (holes bounded by a curve) on the boundary of a simplex.

Sphere

Let us compute $H_2(\Delta)$. There are no boundaries, since $C_3(\Delta) = 0$. In other words, $B_2(\Delta) = 0$. A chain $m_1 BCD + m_2 CAD + m_3 ABD + m_4 ACB$ is a cycle if and only if its boundary is zero, which is

$$
\begin{aligned}
m_1(CD + DB + BC) + m_2(AD + DC + CA) + \\
m_3(BD + DA + AB) + m_4(CB + BA + AC) = \\
(m_3 - m_4)AB + (m_4 - m_2)AC + (m_2 - m_3)AD + \\
(m_1 - m_4)BC + (m_3 - m_1)BD + (m_1 - m_2)CD
\end{aligned}
$$

It is zero if and only if $m_1 = m_2 = m_3 = m_4$. We see that the group of cycles is cyclic, generated by $BCD + CAD + ABD + ACB$. It follows that $H_2(\Delta) = \mathbb{Z}$, i.e., we have one two-dimensional "hole."

In general, if Δ_n is the boundary of an $(n+1)$-dimensional simplex, then $H_0(\Delta_n) = \mathbb{Z}$, $H_n(\Delta_n) = \mathbb{Z}$, and all the remaining homology groups are 0.

The homology groups $H_n(\Delta)$ depend on the topological type of Δ. For example, if Δ is a *triangulation* of a surface S, then $H_n(\Delta)$ depends only on S, and can be denoted $H_n(S)$. For example,s if Δ is the simplicial complex of the surface of an octahedron, or icosahedron, then $H_n(\Delta)$ is the same as $H_n(\Delta)$ for Δ the surface of the tetrahedron, since all of them are triangulations of the sphere.