MATH 416, Modern Algebra II

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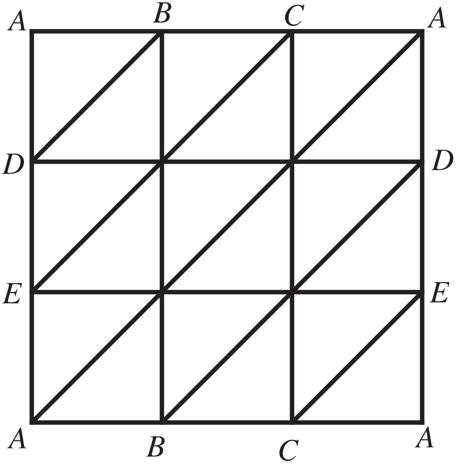
Torus

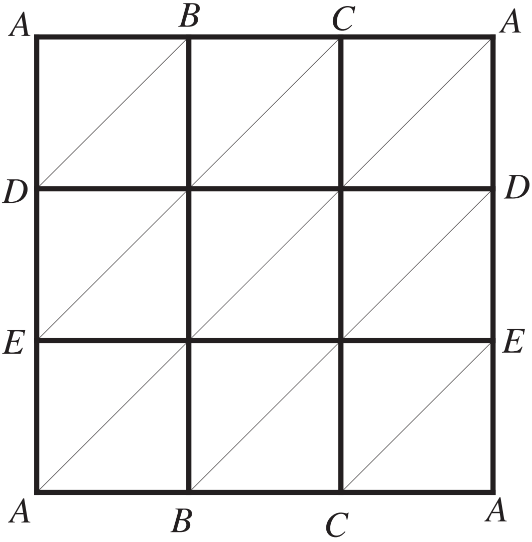
The torus can be realized as the following simplicial complex.



Torus

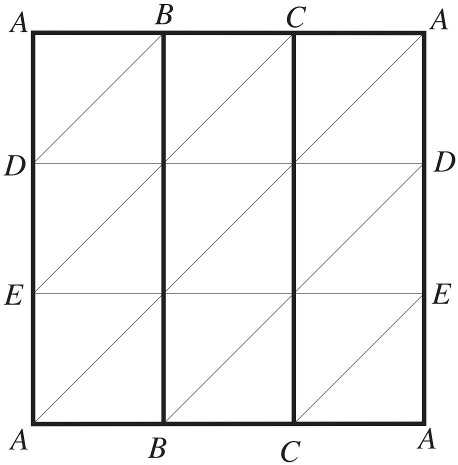
We have $H_0(T) = \mathbb{Z}$, since torus is connected.
Let us compute $H_1(T) = Z_1(T)/B_1(T)$. If we have a cycle containing BD, then we can replace it by $BA + AD$, since their images modulo $B_1(T)$ are equal (the coset $BD + B_1(T)$ is equal to $BA + AD + B_1(T)$, since $BA + AD - BD = BA + AD + DB \in B_1(T)$). Consequently, any cycle can be represented by a *homologous* (i.e., equal modulo $B_1(T)$) cycle consisting only of vertical and horizontal edges of the triangulation.

Torus

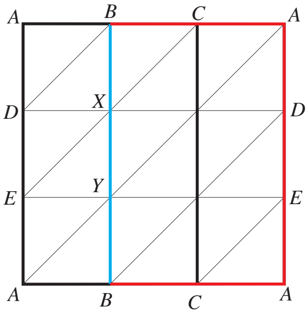


Torus

Similarly, we can replace any "internal" horizontal edge by the sum of three sides of a rectangle, so that we get a chain equal to a linear combination of the edges on the boundary of the big square and vertical edges.



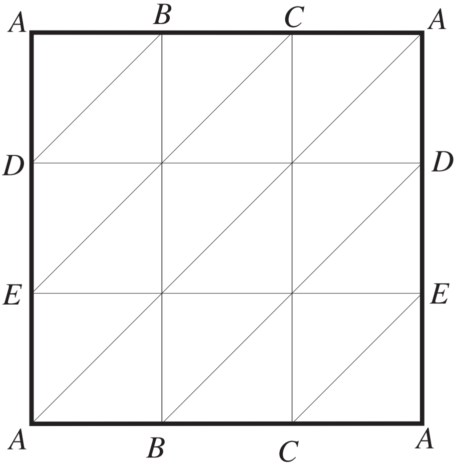
Torus



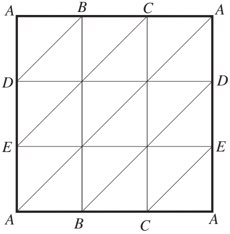
If such a cycle, contains summands $m_1 BX + m_2 XY + m_3 YB$, then its boundary contains $(m_2 - m_1)X + (m_3 - m_2)Y$, so $m_1 = m_2 = m_3$. Then the part $mBX + mXY + mYB$ of the cycle can be replaced by a homologous chain $m(BC + CA + AD + DE + EA + AC + CB)$. We can do the same with the circle passing through C.

Torus

We see that any cycle is homologous to a cycle supported on the two circles represented by the boundary of the square.



Torus



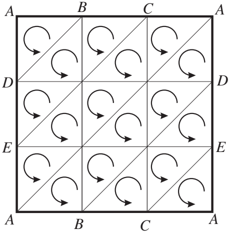
We have

$$\partial(m_1 AB + m_2 BC + m_3 CA + n_1 AD + n_2 DE + n_3 EA) =$$
$$m_1 A - m_1 B + m_2 B - m_2 C + m_3 C - m_3 A + n_1 A - n_1 D + n_2 D - n_2 E + n_3 E - n_3 A =$$
$$(m_1 - m_3 + n_1 - n_3)A + (m_2 - m_1)B + (m_3 - m_2)C + (n_2 - n_1)D + (n_3 - n_2)E$$

It is equal to 0 if and only if $m_1 = m_2 = m_3$ and $n_1 = n_2 = n_3$. (It is the same "figure eight" example.)

Torus

It follows that $H_1(T)$ is generated by the two circles $AB + BC + CA$ and $AD + DE + EA$. Suppose now $m(AB + BC + CA) + n(AD + DE + EA)$ is a boundary of some 2-chain, i.e., of some linear combination of the triangles. Let us orient the triangles the same way.



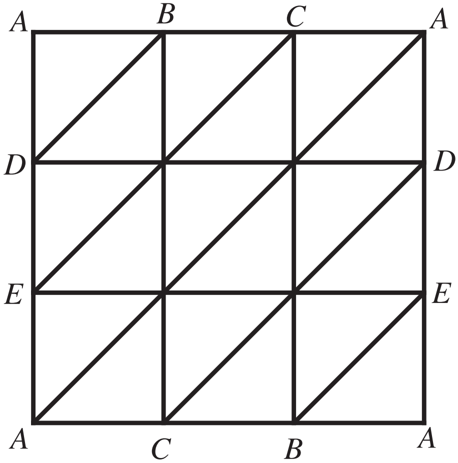
Then the coefficients of any two adjacent triangles must be equal, since their common edge does not appear in the boundary, so the coefficients must cancel. But then also the coefficients of all the remaining edges will also cancel, so $m = n = 0$.

Torus

It follows that $AB + BC + CA$ and $AD + DE + EA$ are independent in $H_1(T)$, hence $H_1(T) = \mathbb{Z}^2$.
A linear combination of triangles is a cycle, i.e., has zero boundary, if and only if all the coefficients are equal (by the same argument as before, since the coefficient of any common edge must be 0). It follows that $Z_2(T)$ is generated by the sum of all triangles, so that $Z_2(T) = \mathbb{Z}$. Since $B_2(T) = 0$, we get $H_2(T) = \mathbb{Z}$.

Klein bottle

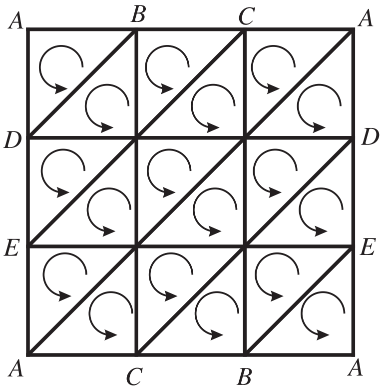
The Klein bottle can be defined as the following simplicial complex K



Klein bottle

By the same argument as for the torus, $H_1(K)$ is generated by the images of the two circles $AB + BC + CA$ and $AD + DE + EA$. We also have that boundary of a 2-chain is of the form
$m(AB + BC + CA) + n(AD + DE + EA)$ if and only if all coefficients of the triangles are equal.

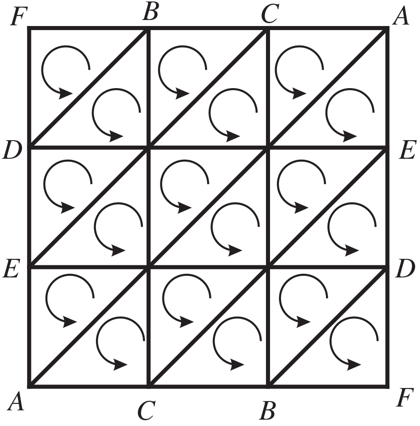
Klein bottle



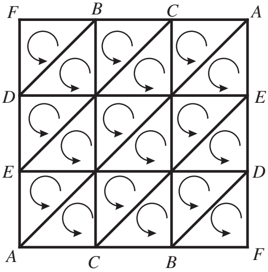
The boundary of the sum of all triangles is $2BA + 2CB + 2AC$. It follows that $H_2(K)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2$. There is *torsion* in the homology.

Projective plane

The *projective plane* $\mathbb{PR}^2$ is obtained by identifying the opposite points of a disc (equivalently, the opposite points of a sphere). It can be realized as the following simplicial complex.



Projective plane



The same arguments show that any cycle is homologous to $m(AC + CB + BF) + n(FD + DE + EA)$. Note, however, that it is a cycle if and only if $m = n$, since the coefficient at F in the boundary is $n - m$. Consequently, $H_1(\mathbb{PR}^2)$ is in this case cyclic generated by $AC + CB + BF + FD + DE + EA$. The boundary of the sum of all triangles is $2(AC + CB + BF + FD + DE + EA)$, which implies that $H_1(\mathbb{PR}^2) = \mathbb{Z}_2$.

Induced maps on the homology

Suppose that $f : X \to Y$ is a continuous map. Let us triangulate X and Y (i.e., transform them into simplicial complexes), and assume that f maps simplices to simplices. (One has sometimes to "deform" f in a continuous way to achieve this.) Then it will map chains to chains, boundaries to boundaries, therefore, it will induce well defined maps $f_* : H_n(X) \to H_n(Y)$.

Example: degree

Let $f : S^1 \to S^1$ be the map $z \mapsto z^3$ on the complex unit circle. Triangulate S^1 by realizing it as a regular 9-gon P_9 with vertices in the solutions of $z^9 = 1$ and as a triangle P_3 with the vertices in the roots of $z^3 = 1$. Then $f : P_9 \to P_3$ will map simplices to simplices. We have $H_1(S^1) = \mathbb{Z}$. The group $H_1(P_9)$ is generated by the sum of all sides of the 9-gon. The group $H_1(P_3)$ is also generated by the sum of the sides of the triangle. f maps a side of P_9 to a side of P_3. It follows that f maps the generator of $H_1(P_9)$ to 3 times the generator of P_3. Therefore, $f_* : H_1(S^1) \longrightarrow H_1(S^1)$ is the map $n \mapsto 3n$ on $\mathbb{Z}$.

Example:degree

In general, every continuous map $f : S^1 \to S^1$ will induce a homomorphism $f_* : H_1(S^1) \to H_1(S^1)$. It is a homomorphism $\mathbb{Z} \to \mathbb{Z}$, hence it is given by $n \mapsto dn$ for some d. The number d tells us how many times f winds one circle around the other and in which direction. For example, if f is not surjective, then $d = 0$.
Similarly, for every sphere S^n a continuous map $f : S^n \to S^n$ induces a map $\mathbb{Z} \to \mathbb{Z}$ in the nth homology $H_n(S^n) \to H_n(S^n)$.