

Student (Print) _____

Last, First Middle

Section _____

Student (Sign) _____

Student ID _____

Instructor _____

MATH 152, Fall 2007
Common Exam 2
Test Form B

Solutions

Instructions:

You may not use notes, books, calculator or computer.

Part I is multiple choice. There is no partial credit.

Mark the Scantron with a #2 pencil. For your own records, also circle your choices in this exam. Scantrons will be collected after 90 minutes and may not be returned.

Part II is work out. Show all your work. Partial credit will be given.

THE AGGIE CODE OF HONOR:

An Aggie does not lie, cheat or steal, or tolerate those who do.

For Dept use Only:	
1-10	/50
11	/10
12	/10
13	/10
14	/10
15	/10
TOTAL	

Part I: Multiple Choice (5 points each)

There is no partial credit.

1. Use the Trapezoid Rule with 3 intervals to approximate $\int_1^7 x^2 dx$.
- a. 59
 - b. 84
 - c. 114
 - d. 118 correct choice
 - e. 168

$$\Delta x = \frac{7-1}{3} = 2 \quad f(x) = x^2$$

$$T_3 = \Delta x \left(\frac{1}{2}f(1) + f(3) + f(5) + \frac{1}{2}f(7) \right) = 2 \left(\frac{1}{2} + 9 + 25 + \frac{1}{2}49 \right) = 118$$

2. If you use the trapezoid rule with 47 intervals to estimate $\int_0^{94} e^{x^3} dx$ your answer will be
- a. too large because e^{x^3} is concave down.
 - b. too large because e^{x^3} is concave up. correct choice
 - c. too small because e^{x^3} is concave up.
 - d. too small because e^{x^3} is increasing.
 - e. too small because e^{x^3} is decreasing.

Since e^{x^3} is concave up, the secant lines are all above the curve.

So the area under the trapezoids is larger than the area under the curve.

3. Find the integrating factor for $\frac{dy}{dx} - \frac{4}{x}y = x^2$
- a. x^{-4} correct choice
 - b. $-4 \ln x$
 - c. x^4
 - d. $4 \ln x$
 - e. $-\frac{1}{3}x^{-3}$

$$P = -\frac{4}{x} \quad \int P dx = -4 \ln x \quad I = e^{-4 \ln x} = x^{-4}$$

4. Compute $\int_0^{\pi/2} \sec^2 \theta d\theta$

- a. -1
- b. 0
- c. 1
- d. 2
- e. diverges correct choice

$$\int_0^{\pi/2} \sec^2 \theta d\theta = [\tan \theta]_0^{\pi/2} = \lim_{x \rightarrow \frac{\pi}{2}^-} \tan \theta - \tan 0 = \infty - 0 = \infty$$

5. Compute $\int_{-1}^4 \frac{1}{x^2} dx$

- a. $-\frac{5}{4}$
- b. 0
- c. $\frac{3}{4}$
- d. $\frac{5}{4}$
- e. diverges correct choice

$$\begin{aligned} \int_{-1}^4 \frac{1}{x^2} dx &= \int_{-1}^0 x^{-2} dx + \int_0^4 x^{-2} dx = \left[\frac{x^{-1}}{-1} \right]_{-1}^0 + \left[\frac{x^{-1}}{-1} \right]_0^4 = \left[-\frac{1}{x} \right]_{-1}^0 + \left[-\frac{1}{x} \right]_0^4 \\ &= \lim_{x \rightarrow 0^-} \left(-\frac{1}{x} \right) - \left(-\frac{1}{-1} \right) + \left(-\frac{1}{4} \right) - \lim_{x \rightarrow 0^+} \left(-\frac{1}{x} \right) = \left(-\frac{1}{0^-} \right) - 1 - \frac{1}{4} - \left(-\frac{1}{0^+} \right) \\ &= \infty - \frac{5}{4} + \infty = \infty \end{aligned}$$

6. The integral $\int_{\pi}^{\infty} \frac{3 + \sin(x)}{x} dx$ is

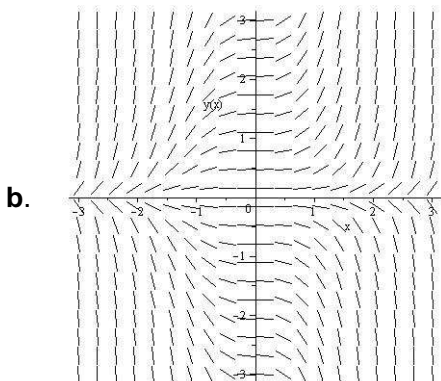
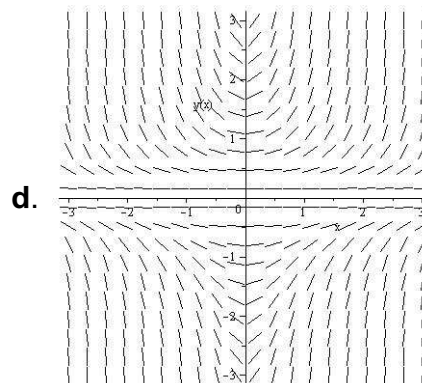
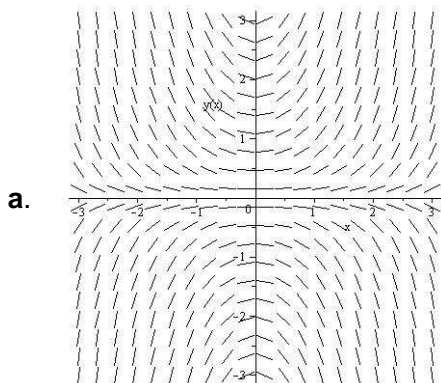
- a. divergent by comparison with $\int_{\pi}^{\infty} \frac{4}{x} dx$.
- b. divergent by comparison with $\int_{\pi}^{\infty} \frac{3}{x} dx$.
- c. divergent by comparison with $\int_{\pi}^{\infty} \frac{2}{x} dx$. correct choice
- d. convergent by comparison with $\int_{\pi}^{\infty} \frac{4}{x} dx$.
- e. convergent by comparison with $\int_{\pi}^{\infty} \frac{2}{x} dx$.

The integrals $\int_{\pi}^{\infty} \frac{4}{x} dx$ and $\int_{\pi}^{\infty} \frac{2}{x} dx$ both diverge because $\int_{\pi}^{\infty} \frac{k}{x} dx = k[\ln x]_{\pi}^{\infty} = \infty$

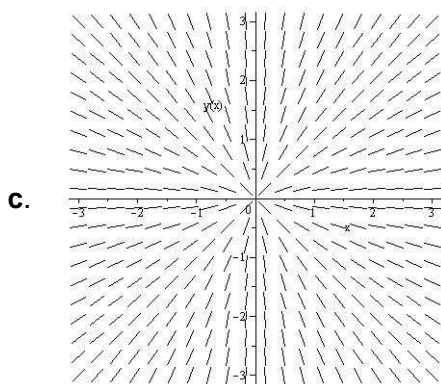
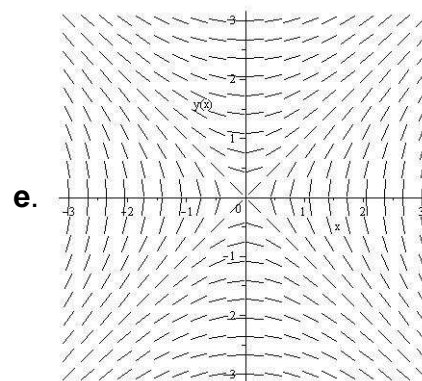
Since $\frac{3 + \sin(x)}{x} \geq \frac{2}{x}$, we have $\int_{\pi}^{\infty} \frac{3 + \sin(x)}{x} dx \geq \infty$ and so diverges.

Since $\frac{3 + \sin(x)}{x} \leq \frac{4}{x}$, we have $\int_{\pi}^{\infty} \frac{3 + \sin(x)}{x} dx \leq \infty$ and cannot conclude anything.

7. Which of the following diagrams is the direction field of the differential equation $\frac{dy}{dx} = x^2y$?



correct choice



The slope is $m = x^2y$.
 This is positive when $y > 0$,
 negative when $y < 0$ and
 zero when $y = 0$. Only plot
b has positive slope in the
 second quadrant.

8. Suppose the function $y = f(x)$ is the solution of $\frac{dy}{dx} = x^2 - y^2$ satisfying the initial condition $f(1) = 2$. Find $f'(1)$.

- a. -3 correct choice
- b. -2
- c. -1
- d. 1
- e. 2

As a solution, $y = f(x)$ satisfies $\frac{df}{dx} = x^2 - f(x)^2$. So $f'(1) = 1^2 - f(1)^2 = 1^2 - 2^2 = -3$

9. Find the length of the curve $x = 2t^2$, and $y = t^3$ for $0 \leq t \leq 1$.

- a. $\frac{1}{3}$
- b. $\frac{61}{27}$ correct choice
- c. 7
- d. 108
- e. 732

As a t -integral, the differential of arc length is

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \sqrt{(4t)^2 + (3t^2)^2} dt = \sqrt{16t^2 + 9t^4} dt = t\sqrt{16 + 9t^2} dt$$

So the arc length is

$$L = \int ds = \int_0^1 t\sqrt{16 + 9t^2} dt \quad u = 16 + 9t^2 \quad du = 18t dt$$

$$L = \frac{1}{18} \int_{16}^{25} \sqrt{u} du = \left[\frac{1}{18} \frac{2u^{3/2}}{3} \right]_{16}^{25} = \frac{1}{27} (125 - 64) = \frac{61}{27}$$

10. The curve $y = \sin x$ for $0 \leq x \leq \pi$ is rotated about the x -axis. Which formula gives the area of the resulting surface?

- a. $\int_0^\pi \pi \cos x \sqrt{1 + \sin^2 x} dx$
- b. $\int_0^\pi 2\pi \cos x \sqrt{1 + \sin^2 x} dx$
- c. $\int_0^\pi \pi \sin x \sqrt{1 + \cos^2 x} dx$
- d. $\int_0^\pi 2\pi \sin x \sqrt{1 + \cos^2 x} dx$ correct choice
- e. $\int_0^\pi \pi \sin x dx$

As an x -integral, the differential of arc length is

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \sqrt{1 + \cos^2 x} dx$$

The radius of revolution is $r = y = \sin x$. So the surface area is

$$A = \int_0^\pi 2\pi r ds = \int_0^\pi 2\pi \sin x \sqrt{1 + \cos^2 x} dx$$

Part II: Work Out (10 points each)

Show all your work. Partial credit will be given.

11. Compute $\int_1^{\infty} \frac{\ln x}{x^2} dx$

Integrate by parts: $u = \ln x \quad dv = \frac{1}{x^2} dx$
 $du = \frac{1}{x} dx \quad v = -\frac{1}{x}$

$$\int_1^{\infty} \frac{\ln x}{x^2} dx = -\frac{\ln x}{x} + \int \frac{1}{x^2} dx = \left[-\frac{\ln x}{x} - \frac{1}{x} \right]_1^{\infty} = \lim_{x \rightarrow \infty} \left(-\frac{\ln x}{x} - \frac{1}{x} \right) - \left(-\frac{\ln 1}{1} - \frac{1}{1} \right)$$

$$= -\lim_{x \rightarrow \infty} \left[\frac{\ln x}{x} \right] + 1 \stackrel{L'H}{=} -\lim_{x \rightarrow \infty} \left[\frac{\frac{1}{x}}{1} \right] + 1 = 1$$

12. Simpson's Error Formula says:

The error E_S in approximating $\int_a^b f(x) dx$ with n intervals satisfies

$$|E_S| \leq \frac{K(b-a)^5}{180n^4} \quad \text{where } K \geq \max |f^{(4)}(x)| \text{ for } a \leq x \leq b.$$

Use this formula to estimate the error in approximating $\int_1^4 \frac{1}{x} dx$ using $n = 30$ intervals.

$$b - a = 4 - 1 = 3$$

$$f = \frac{1}{x} = x^{-1} \quad f' = -x^{-2} \quad f'' = 2x^{-3} \quad f''' = -6x^{-4} \quad f^{(4)} = 24x^{-5}$$

The largest value of $24x^{-5}$ occurs when $x = 1$. So the smallest K is $K = 24$.

$$|E_S| < \frac{K(b-a)^5}{180n^4} = \frac{24(3)^5}{180(30)^4} = \frac{24 \cdot 3}{180(10)^4} = \frac{8}{20(10)^4} = \frac{4}{10(10)^4} = 4 \times 10^{-5}$$

13. Find the centroid of the region in the first quadrant bounded by $y = 2\sqrt{x}$, the x -axis and $x = 4$.

$$A = \int_0^4 f(x) dx = \int_0^4 2\sqrt{x} dx = \left[\frac{4x^{3/2}}{3} \right]_0^4 = \frac{4 \cdot 4^{3/2}}{3} = \frac{32}{3}$$

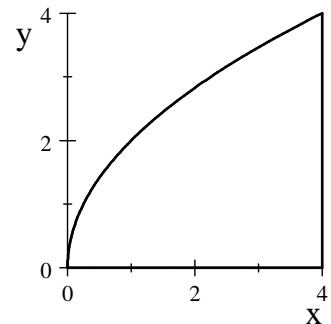
$$M_y = \int_0^4 xf(x) dx = \int_0^4 x \cdot 2\sqrt{x} dx = \left[\frac{4x^{5/2}}{5} \right]_0^4 = \frac{4 \cdot 4^{5/2}}{5} = \frac{128}{5}$$

$$\bar{x} = \frac{M_y}{A} = \frac{3}{32} \frac{128}{5} = \frac{12}{5}$$

$$M_x = \int_0^4 \frac{1}{2} f(x)^2 dx = \int_0^4 \frac{1}{2} 4x dx = \left[x^2 \right]_0^4 = 16$$

$$\bar{y} = \frac{M_x}{A} = \frac{3}{32} 16 = \frac{3}{2}$$

$$(x, y) = \left(\frac{12}{5}, \frac{3}{2} \right)$$



14. Consider the differential equation: $\frac{dy}{dx} = \frac{xy}{x^2 + 16}$

a. Find the general solution:

METHOD 1: The equation is separable. Separate and integrate.

$$\int \frac{dy}{y} = \int \frac{x}{x^2 + 16} dx \quad \ln|y| = \frac{1}{2} \ln(x^2 + 16) + C = \ln \sqrt{x^2 + 16} + C$$

Solve for y :

$$|y| = e^C \sqrt{x^2 + 16} \quad y = A \sqrt{x^2 + 16} \quad \text{where } A = \pm e^C$$

METHOD 2: The equation is linear: Put in standard form. $\frac{dy}{dx} - \frac{x}{x^2 + 16}y = 0$

$$P(x) = -\frac{x}{x^2 + 16} \quad \int P(x) dx = -\int \frac{x}{x^2 + 16} dx = -\frac{1}{2} \ln(x^2 + 16)$$

Find the integrating factor. $I = e^{\int P(x) dx} = e^{-\frac{1}{2} \ln(x^2 + 16)} = (x^2 + 16)^{-1/2} = \frac{1}{\sqrt{x^2 + 16}}$

Multiply the standard equation by $I(t)$. $\frac{1}{\sqrt{x^2 + 16}} \frac{dy}{dx} - \frac{x}{(x^2 + 16)^{3/2}} y = 0$

Recognize the left side as a derivative of a product. $\frac{d}{dx} \left(\frac{1}{\sqrt{x^2 + 16}} y \right) = 0$

Integrate and solve. $\frac{1}{\sqrt{x^2 + 16}} y = A \quad y = A \sqrt{x^2 + 16}$

b. Find the solution satisfying the initial condition: $y(0) = 8$

Substitute $x = 0$ and $y = 8$:

$$8 = A \sqrt{0 + 16} = 4A \quad A = 2 \quad y = 2 \sqrt{x^2 + 16}$$

c. Find $y(3)$:

$$y(3) = 2 \sqrt{3^2 + 16} = 10$$

15. A salt water fish tank contains 20 liters of water with 800 grams of salt. In order to reduce the salt concentration, you pour in salt water with a concentration of 15 grams of salt per liter at 2 liters per minute. You keep the tank well mixed and drain the mixture at 2 liters per minute. Let $S(t)$ be the amount of salt (in grams) in the tank at time t (in minutes).

a. Write the differential equation and initial condition for $S(t)$.

$$\frac{dS}{dt} = 15 \frac{g}{l} \cdot 2 \frac{l}{min} - \frac{S}{20} \cdot 2 \frac{l}{min} \quad \text{or} \quad \frac{dS}{dt} = 30 - \frac{1}{10} S$$

$$S(0) = 800$$

b. Solve the initial value problem for $S(t)$.

METHOD 1: The equation is separable. Separate and integrate.

$$\int \frac{dS}{30 - \frac{1}{10} S} = \int dt \quad -10 \ln \left| 30 - \frac{1}{10} S \right| = t + C$$

$$\text{Solve.} \quad \ln \left| 30 - \frac{1}{10} S \right| = -\frac{t}{10} - \frac{C}{10} \quad \left| 30 - \frac{1}{10} S \right| = e^{-C/10} e^{-t/10}$$

$$30 - \frac{1}{10} S = A e^{-t/10} \quad \text{where } A = \pm e^{-C/10} \quad \frac{1}{10} S = 30 - A e^{-t/10}$$

$$S = 300 - 10A e^{-t/10}$$

Use the initial condition. At $t = 0$, we have $S = 800$

$$800 = 300 - 10A \quad 10A = -500 \quad A = -50$$

$$\text{Substitute back.} \quad S = 300 + 500 e^{-t/10}$$

METHOD 2: The equation is linear: Put in standard form. $\frac{dS}{dt} + \frac{1}{10} S = 30$

Find the integrating factor. $I = \exp\left(\int \frac{1}{10} dt\right) = e^{t/10}$

Multiply the standard equation by $I(t)$. $e^{t/10} \frac{dS}{dt} + \frac{1}{10} e^{t/10} S = 30 e^{t/10}$

Recognize the left side as a derivative of a product. $\frac{d}{dt}(e^{t/10} S) = 30 e^{t/10}$

Integrate and solve. $e^{t/10} S = \int 30 e^{t/10} dt = 300 e^{t/10} + C \quad S = 300 + C e^{-t/10}$

Use the initial condition. At $t = 0$, we have $S = 800$.

$$800 = 300 + C \quad C = 500$$

$$\text{Substitute back.} \quad S = 300 + 500 e^{-t/10}$$

c. After how many minutes will the amount of salt in the tank drop to 400 grams?

Set $S = 400$ and solve for t :

$$400 = 300 + 500 e^{-t/10} \quad 500 e^{-t/10} = 100 \quad e^{-t/10} = \frac{1}{5} \quad \frac{-t}{10} = \ln\left(\frac{1}{5}\right)$$

$$t = -10 \ln\left(\frac{1}{5}\right) \quad t = 10 \ln 5$$