Name________________________ Sec______ ID______________

1-11	/66
12	/12
13	/15
14	/10
Total	/103

MATH 152 Final Exam Fall 2007

Sections 513 - 515 Solutions P. Yasskin

Multiple Choice: (6 points each)

1. A plate is bounded by the curves $y = x^2$, $y = -x^2$ and $x = 2$ measured in meters. Find the total mass of the plate if the surface density is $\rho = 3$ kg/m^2.

a. $\frac{8}{3}$

b. $\frac{16}{3}$

c. $\frac{32}{3}$

d. 8

e. 16 correct choice

$$M = \rho\int_0^2 (f-g)\,dx = 3\int_0^2 (x^2 - -x^2)\,dx = 3\left[\frac{2x^3}{3}\right]_0^2 = 16$$

2. A plate is bounded by the curves $y = x^2$, $y = -x^2$ and $x = 2$ measured in meters. Find the center of mass of the plate if the surface density is $\rho = 3$ kg/m^2.

a. $\left(\frac{3}{4}, 0\right)$

b. $\left(\frac{3}{4}, \frac{1}{10}\right)$

c. $\left(\frac{3}{4}, \frac{6}{5}\right)$

d. $\left(\frac{3}{2}, 0\right)$ correct choice

e. $\left(\frac{3}{2}, \frac{6}{5}\right)$

$$M_y = \rho\int_0^2 x(f-g)\,dx = 3\int_0^2 x(x^2 - -x^2)\,dx = 3\left[\frac{2x^4}{4}\right]_0^2 = 24$$

$\bar{x} = \frac{M_y}{M} = \frac{24}{16} = \frac{3}{2}$ $\bar{y} = 0$ by symmetry.

3. Compute $\int x^2 e^{2x}\,dx$.

a. $\frac{x^2}{2}e^{2x} - \frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + C$

b. $\frac{x^2}{2}e^{2x} - \frac{x}{2}e^{2x} + \frac{1}{4}e^{2x} + C$ correct choice

c. $\frac{x^2}{2}e^{2x} - \frac{x}{2}e^{2x} - \frac{1}{2}e^{2x} + C$

d. $\frac{x^2}{2}e^{2x} - \frac{x}{2}e^{2x} + \frac{1}{2}e^{2x} + C$

e. $\frac{x^2}{2}e^{2x} - \frac{x}{4}e^{2x} - \frac{1}{4}e^{2x} + C$

$u = x^2 \qquad dv = e^{2x}\,dx$

$du = 2x\,dx \qquad v = \frac{1}{2}e^{2x}$

$$\int x^2 e^{2x}\,dx = \frac{x^2}{2}e^{2x} - \int x e^{2x}\,dx$$

$u = x \qquad dv = e^{2x}\,dx$

$du = dx \qquad v = \frac{1}{2}e^{2x}$

$$\int x^2 e^{2x}\,dx = \frac{x^2}{2}e^{2x} - \left(\frac{x}{2}e^{2x} - \frac{1}{2}\int e^{2x}\,dx\right) = \frac{x^2}{2}e^{2x} - \frac{x}{2}e^{2x} + \frac{1}{4}e^{2x} + C$$

4. The parametric curve $x = 2t^2$ $y = t^3$ for $0 \le t \le 1$ is rotated about the y-axis. Which integral gives the area of the surface swept out?

a. $2\pi \int_0^1 t\sqrt{16 + 9t^2}\,dt$

b. $2\pi \int_0^1 t^3\sqrt{16 + 9t^2}\,dt$

c. $4\pi \int_0^1 t^3\sqrt{16 + 9t^2}\,dt$ correct choice

d. $2\pi \int_0^1 t^4\sqrt{16 + 9t^2}\,dt$

e. $4\pi \int_0^1 t^4\sqrt{16 + 9t^2}\,dt$

$$A = \int 2\pi r\,ds = \int_0^1 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^1 2\pi 2t^2\sqrt{(4t)^2 + (3t^2)^2}\,dt$$

$$= 4\pi \int_0^1 t^2\sqrt{16t^2 + 9t^4}\,dt = 4\pi \int_0^1 t^3\sqrt{16 + 9t^2}\,dt$$

5. Which term appears in the partial fraction expansion of $\frac{3x^2 - 4x - 20}{(x-2)^2(x^2+4)}$?

- **a.** $\frac{-2}{(x-2)^2}$ correct choice
- **b.** $\frac{1}{(x-2)^2}$
- **c.** $\frac{2}{(x-2)^2}$
- **d.** $\frac{-2}{(x-2)}$
- **e.** $\frac{1}{(x-2)}$

$$\frac{3x^2 - 4x - 20}{(x-2)^2(x^2+4)} = \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{Cx+D}{(x^2+4)}$$

$$3x^2 - 4x - 20 = A(x-2)(x^2+4) + B(x^2+4) + (Cx+D)(x-2)^2$$

$$x = 2: \quad 12 - 8 - 20 = 0 + B(8) + 0 \qquad -16 = 8B \qquad B = -2$$

6. Find the solution of the differential equation $x\frac{dy}{dx} = 2y + x^3$ satisfying the initial condition $y(1) = 2$.

- **a.** $y = x^3 - \frac{7}{4}x^2$
- **b.** $y = x^3 + x^2$ correct choice
- **c.** $y = 2x^3 - x^2$
- **d.** $y = \frac{9}{5x^2} + \frac{1}{5}x^3$
- **e.** $y = \frac{5}{3x^2} + \frac{1}{3}x^3$

Put the equation in standard linear form: $\frac{dy}{dx} - \frac{2}{x}y = x^2$

The integrating factor is: $I = e^{\int -\frac{2}{x}\,dx} = e^{-2\ln x} = x^{-2} = \frac{1}{x^2}$

Multiply by the integrating factor: $\frac{1}{x^2}\frac{dy}{dx} - \frac{2}{x^3}y = 1$

Rewrite and integrate: $\frac{d}{dx}\left(\frac{1}{x^2}y\right) = 1 \qquad \frac{1}{x^2}y = \int 1\,dx = x + C$

Apply the initial conditions $x = 1$, $y = 2$: $\quad 2 = 1 + C \qquad C = 1$

Substitute back and solve for y: $\quad \frac{1}{x^2}y = x + 1 \qquad y = x^3 + x^2$

7. The region in the first quadrant bounded by the curves $y = x^2$, $y = 0$ and $x = 2$ is rotated about the y-axis. Find the volume of the solid swept out.

a. $\frac{32}{5}\pi$

b. $\frac{64}{5}\pi$

c. 8π correct choice

d. 4π

e. 2π

This is a x-integral. A vertical slice rotates into a cylinder of radius $r = x$ and height $h = x^2$.

So the volume is

$$V = \int_0^2 2\pi r h\, dx = \int_0^2 2\pi\, x x^2\, dx = 2\pi \frac{x^4}{4}\Big|_0^2 = 8\pi$$

8. If you approximate $f(x) = \ln(x)$ on the interval $\left[\frac{1}{2}, \frac{3}{2}\right]$ by its 3^{rd} degree Taylor polynomial centered at $x = 1$, namely $T_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$, then the Taylor Remainder Theorem says the error $|R_3(x)|$ is less than

Taylor Remainder Theorem:

If $T_n(x)$ is the n^{th} degree Taylor polynomial for $f(x)$ centered at $x = a$ then there is a number c between a and x such that the remainder is

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{(n+1)}$$

a. 1

b. $\frac{1}{2}$

c. $\frac{1}{4}$ correct choice

d. $\frac{1}{8}$

e. 0

Here $a = 1$, and $n = 3$, and $|R_3(x)| = \dfrac{|f^{(4)}(c)|}{4!}|x-1|^4$

We need the largest values of $|f^{(4)}(c)|$ and $|x-1|^4$ for c and x in $\left[\frac{1}{2}, \frac{3}{2}\right]$.

$f'(x) = \frac{1}{x} = x^{-1}$ $\quad f''(x) = -x^{-2}$ $\quad f'''(x) = 2x^{-3}$ $\quad f^{(4)}(x) = -6x^{-4}$

The max of $|f^{(4)}(c)| = \frac{6}{c^4}$ occurs at $c = \frac{1}{2}$ and is $M = 6 \cdot 2^4$.

The max of $|x-1|^4$ occurs at $\frac{1}{2}$ or $\frac{3}{2}$ and is $\frac{1}{2^4}$.

So $|R_n(x)| \le \dfrac{6 \cdot 2^4}{4!}\dfrac{1}{2^4} = \dfrac{6}{24} = \dfrac{1}{4}$

9. Compute $\lim\limits_{x\to 0} \dfrac{\sin(x^2) - x^2}{e^{\left(x^3\right)} - 1 - x^3}$.

a. $-\dfrac{1}{6}$

b. $\dfrac{1}{6}$

c. $\dfrac{2}{3}$

d. $-\dfrac{1}{3}$ correct choice

e. $\dfrac{1}{3}$

$$\sin(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots \qquad \sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots$$

$$e^{(t)} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots \qquad e^{\left(x^3\right)} = 1 + x^3 + \frac{x^6}{2!} + \frac{x^9}{3!} + \cdots$$

$$\lim_{x\to 0} \frac{\sin(x^2) - x^2}{e^{\left(x^3\right)} - 1 - x^3} = \lim_{x\to 0} \frac{-\dfrac{x^6}{3!} + \dfrac{x^{10}}{5!} - \cdots}{\dfrac{x^6}{2!} + \dfrac{x^9}{3!} + \cdots} = \lim_{x\to 0} \frac{-\dfrac{1}{3!} + \dfrac{x^4}{5!} - \cdots}{\dfrac{1}{2!} + \dfrac{x^3}{3!} + \cdots} = -\frac{2!}{3!} = -\frac{1}{3}$$

10. Find the volume of the parallelepiped with edge vectors

$$\vec{a} = (-2,2,1), \qquad \vec{b} = (3,2,4) \qquad \text{and} \qquad \vec{c} = (1,-2,3)$$

a. -26

b. 26

c. $\sqrt{26}$

d. -46

e. 46 correct choice

$$V = \left\| \begin{matrix} -2 & 2 & 1 \\ 3 & 2 & 4 \\ 1 & -2 & 3 \end{matrix} \right\| = |-2(6+8) - 2(9-4) + 1(-6-2)| = |-28 - 10 - 8| = |-46| = 46$$

11. If $\vec{u}$ points Down and $\vec{v}$ points North West, in which direction does $\vec{u} \times \vec{v}$ point?

a. North East correct choice

b. South

c. South West

d. South East

e. Up

Hold your right hand with the fingers pointing down and the palm facing left of forward. Then you thumb points right of forward.

Work Out: (Points indicated. Part credit possible.)

12. (12 points) Compute $\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx$

$x = 2\sin\theta \qquad dx = 2\cos\theta\,d\theta$

$x = 0 \quad @ \quad \sin\theta = 0$ or $\theta = 0 \qquad x = 1 \quad @ \quad \sin\theta = \frac{1}{2}$ or $\theta = \frac{\pi}{6}$

$$\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx = \int_0^{\pi/6} \frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}} 2\cos\theta\,d\theta = \int_0^{\pi/6} 4\sin^2\theta\,d\theta = 4\int_0^{\pi/6} \frac{1-\cos 2\theta}{2}\,d\theta$$

$$= 2\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/6} = 2\left[\frac{\pi}{6} - \frac{1}{2}\sin\frac{\pi}{3}\right] = \frac{\pi}{3} - \frac{\sqrt{3}}{2}$$

13. (15 points) A cone of radius 4 and height 6 and vertex down is filled with water up to height 2. Find the work done to pump the water out the top. Give your answer as a multiple of ρg.

The silce at height y is a circle. So its area is $A = \pi r^2$.

The radius is found using similar triangles: $\frac{r}{y} = \frac{4}{6}$ or $r = \frac{2}{3}y$

So the area is $A = \pi\left(\frac{2}{3}y\right)^2 = \frac{4}{9}\pi y^2$

and the volume of the slice of thickness dy is $dV = \frac{4}{9}\pi y^2\,dy$.

This slice is lifted a distance $D = 6 - y$. So the work is

$$W = \int \rho g D\,dV = \int_0^2 \rho g(6-y)\,\frac{4}{9}\pi y^2\,dy = \frac{4}{9}\rho g\pi \int_0^2 (6y^2 - y^3)\,dy$$

$$= \frac{4}{9}\rho g\pi\left[2y^3 - \frac{y^4}{4}\right]_0^2 = \frac{4}{9}\rho g\pi(16-4) = \frac{16}{3}\rho g\pi$$

14. (10 points) Start from the Maclaurin series: $\dfrac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$

a. (6 pt) Find the Maclaurin series for $\dfrac{1}{(1+x)^2}$.

HINT: Differentiate both sides of the given series.

$$\frac{d}{dx}\left(\frac{1}{1+x}\right) = \frac{-1}{(1+x)^2} \qquad \frac{d}{dx}\left(\sum_{n=0}^{\infty}(-1)^n x^n\right) = \sum_{n=0}^{\infty}(-1)^n\, n\, x^{n-1}$$

$$\frac{1}{(1+x)^2} = -\sum_{n=0}^{\infty}(-1)^n\, n\, x^{n-1}$$ You can stop here.

$$= \sum_{n=0}^{\infty}(-1)^{n-1}\, n\, x^{n-1} = \sum_{n=1}^{\infty}(-1)^{n-1}\, n\, x^{n-1} = \sum_{k=0}^{\infty}(-1)^k\,(k+1)\,x^k$$

b. (4 pt) Compute $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n}{2^{n-1}}$.

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n}{2^{n-1}} = \sum_{n=1}^{\infty}(-1)^{n-1}\, n\left(\frac{1}{2}\right)^{n-1} = \frac{1}{\left(1+\frac{1}{2}\right)^2} = \frac{1}{\left(\frac{3}{2}\right)^2} = \frac{4}{9}$$