MATH 172 FINAL EXAM Spring 1999
Section 504 Solutons P. Yasskin

1. If $\vec{a} = (2,0,1)$ and $\vec{b} = (-1,2,1)$ then $2\vec{a} - 3\vec{b} =$

a. $(1,6,5)$
b. $(1,-6,5)$
c. $(7,6,-1)$
d. $(7,-6,-1)$ correctchoice
e. $(-1,6,-5)$

$$2\vec{a} - 3\vec{b} = 2(2,0,1) - 3(-1,2,1) = (4,0,2) + (3,-6,-3) = (7,-6,-1)$$

2. If $\vec{a} = (2,0,1)$ and $\vec{b} = (-1,2,1)$ then $\vec{a} \times \vec{b} =$

a. $(-2,-3,4)$ correctchoice
b. $(-2,3,4)$
c. $(-2,0,1)$
d. $(4,0,-1)$
e. $(4,3,-2)$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 0 & 1 \\ -1 & 2 & 1 \end{vmatrix} = \hat{\imath}(0-2) - \hat{\jmath}(2--1) + \hat{k}(4-0) = (-2,-3,4)$$

3. If $\vec{a} = (2,0,1)$ and $\vec{b} = (-1,2,1)$ are two edges of a triangle, find the area of the triangle.

a. $\sqrt{3}$
b. $\frac{3}{2}$
c. $\frac{1}{2}\sqrt{29}$ correctchoice
d. $\frac{29}{2}$
e. $\sqrt{29}$

$$A = \frac{1}{2}\left|\vec{a} \times \vec{b}\right| = \frac{1}{2}\sqrt{4+9+16} = \frac{1}{2}\sqrt{29}$$

4. If $\vec{u} = \left(\sqrt{2}, -1, 1\right)$ and $\vec{v} = (0, 1, -1)$ then the angle between $\vec{u}$ and $\vec{v}$ is $\theta =$

- **a.** $30°$
- **b.** $45°$
- **c.** $60°$
- **d.** $120°$
- **e.** $135°$ correctchoice

$\vec{u} \bullet \vec{v} = \left(\sqrt{2}\right)(0) + (-1)(1) + (1)(-1) = -2 \qquad |\vec{u}| = \sqrt{2+1+1} = 2 \qquad |\vec{v}| = \sqrt{1+1} = \sqrt{2}$

$\cos\theta = \dfrac{\vec{u} \bullet \vec{v}}{|\vec{u}||\vec{v}|} = \dfrac{-2}{2\sqrt{2}} = \dfrac{-1}{\sqrt{2}} \qquad \theta = 135°$

5. The series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(x-2)^n$ is the Taylor series about $x = 2$ for

(Hint: Just sum the series.)

- **a.** $\dfrac{1}{x}$ correctchoice
- **b.** $\dfrac{2}{x}$
- **c.** $\dfrac{1}{x-2}$
- **d.** $\dfrac{2}{x-2}$
- **e.** $\dfrac{1}{2(x-2)}$

Geometric series: ratio = $\dfrac{-(x-2)}{2}$ first term = $\dfrac{(-1)^0}{2^1}(x-2)^0 = \dfrac{1}{2}$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(x-2)^n = \frac{\frac{1}{2}}{1 - \frac{-(x-2)}{2}} \bullet \frac{2}{2} = \frac{1}{2+(x-2)} = \frac{1}{x}$$

6. Find the radius of convergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(x-2)^n$.

- **a.** 0
- **b.** 1
- **c.** 2 correctchoice
- **d.** 4
- **e.** 8

Ratio Test: $a_n = \dfrac{(-1)^n}{2^{n+1}}(x-2)^n \qquad a_{n+1} = \dfrac{(-1)^{n+1}}{2^{n+2}}(x-2)^{n+1}$

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{2^{n+2}}\,\frac{2^{n+1}}{(x-2)^n}\right| = \lim_{n\to\infty}\left|\frac{x-2}{2}\right| = \left|\frac{x-2}{2}\right| < 1$$

$|x-2| < 2$ Radius of Convergence $= 2$

7. The series $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n!}$ is

a. Divergent by the n^{th} term Divergence Test
b. Convergent by the Ratio Test correctchoice
c. Divergent by the Ratio Test
d. Convergent by the Integral Test
e. Divergent by the Integral Test

Ratio Test: $a_n = \frac{\sqrt{n+1}}{n!} \qquad a_{n+1} = \frac{\sqrt{n+2}}{(n+1)!}$

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\sqrt{n+2}}{(n+1)!}\frac{n!}{\sqrt{n+1}}\right| = \lim_{n\to\infty}\left|\frac{\sqrt{n+2}}{(n+1)\sqrt{n+1}}\right| = 0 < 1$$

Convergent

8. Compute $\lim_{x\to 0}\frac{\sin(2x) - 2x + \frac{8x^3}{3}}{x^5}$

a. $-\infty$
b. 0
c. $\frac{4}{3}$
d. $\frac{2}{5!}$
e. ∞ correctchoice

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \qquad \sin(2x) = 2x - \frac{8x^3}{6} + \frac{32x^5}{120} - \cdots$$

$$\lim_{x\to 0}\frac{\sin(2x) - 2x + \frac{8x^3}{3}}{x^5} = \lim_{x\to 0}\frac{2x - \frac{8x^3}{6} + \frac{32x^5}{120} - \cdots - 2x + \frac{8x^3}{3}}{x^5} = \lim_{x\to 0}\frac{\frac{8x^3}{6} + \frac{32x^5}{120} - \cdots}{x^5}$$

$$= \lim_{x\to 0}\left[\frac{8}{6}x^{-2} + \frac{32}{120} - \cdots\right] = \infty$$

9. Compute $\int (2x^3 - 1)\sin(x^4 - 2x)\, dx$

a. $2\cos(x^4 - 2x) + C$
b. $-2\cos(x^4 - 2x) + C$
c. $\frac{1}{2}\cos(x^4 - 2x) + C$
d. $-\frac{1}{2}\cos(x^4 - 2x) + C$ correctchoice
e. $\cos(x^4 - 2x) + C$

$$u = x^4 - 2x \qquad du = (4x^3 - 2)dx \qquad \frac{1}{2}du = (2x^3 - 1)dx$$

$$\int (2x^3 - 1)\sin(x^4 - 2x)\, dx = \frac{1}{2}\int \sin u\, du = -\frac{1}{2}\cos u + C = -\frac{1}{2}\cos(x^4 - 2x) + C$$

10. Compute $\int_0^2 \frac{x^2}{\sqrt{4-x^2}}\,dx$

a. 1
b. 2
c. $\frac{\pi}{4}$
d. π correctchoice
e. 2π

$x = 2\sin\theta \qquad dx = 2\cos\theta\,d\theta \qquad \sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = 2\sqrt{1-\sin^2\theta} = 2\cos\theta$

$$\int_0^2 \frac{x^2}{\sqrt{4-x^2}}\,dx = \int_0^{\pi/2} \frac{4\sin^2\theta}{2\cos\theta}\,2\cos\theta\,d\theta = \int_0^{\pi/2} 4\sin^2\theta\,d\theta = 4\int_0^{\pi/2} \frac{1-\cos(2\theta)}{2}\,d\theta$$

$$= 2\left[\theta - \frac{\sin(2\theta)}{2}\right]_0^{\pi/2} = 2\left[\frac{\pi}{2}\right] = \pi$$

11. Compute $\int_0^{\pi/2} \sin^6 x\cos^3 x\,dx$

a. $\frac{1}{63}$
b. $\frac{2}{63}$ correctchoice
c. $\frac{1}{21}$
d. $\frac{4}{63}$
e. $\frac{2}{21}$

$$\int_0^{\pi/2} \sin^6 x\cos^3 x\,dx = \int_0^{\pi/2} \sin^6 x\left(1-\sin^2 x\right)\cos x\,dx \qquad u = \sin x \qquad du = \cos x\,dx$$

$$= \int_0^1 u^6(1-u^2)\,du = \left[\frac{u^7}{7} - \frac{u^9}{9}\right]_0^1 = \left[\frac{1}{7} - \frac{1}{9}\right] = \frac{2}{63}$$

12. $\int_1^\infty \frac{1}{x+e^{2x}}\,dx$ is

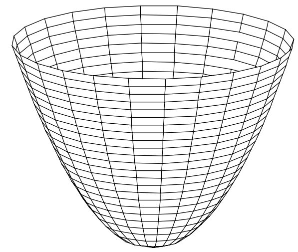
a. Convergent by comparison to $\int_1^\infty \frac{1}{e^{2x}}\,dx$ correctchoice
b. Divergent by comparison to $\int_1^\infty \frac{1}{e^{2x}}\,dx$
c. Convergent by comparison to $\int_1^\infty \frac{1}{x}\,dx$
d. Divergent by comparison to $\int_1^\infty \frac{1}{x}\,dx$

For large x, e^{2x} is larger than x. So we compare to $\int_1^\infty \frac{1}{e^{2x}}\,dx$ which is convergent.

$\int_1^\infty \frac{1}{e^{2x}}\,dx = \left[\frac{e^{-2x}}{-2}\right]_1^\infty = 0 + \frac{e^{-2}}{2} = \frac{e^{-2}}{2}$ which is convergent.

Since $\frac{1}{x+e^{2x}} < \frac{1}{e^{2x}}$ the original series is also convergent.

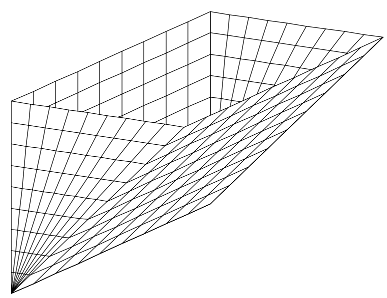
13. The basket shown at the right is 9 in tall. Its horizontal cross sections are circles whose radius is given by $r = 2\sqrt{y}$ where y is the height from the bottom. Find the volume of the basket.



a. 216π
b. 162π correctchoice
c. 108π
d. 81π
e. $\dfrac{81\pi}{2}$

$$V = \int_0^9 A(y)\, dy = \int_0^9 \pi\left(2\sqrt{y}\right)^2 dy = 4\pi \int_0^9 y\, dy = 4\pi\left[\frac{y^2}{2}\right]_0^9 = 162\pi$$

14. A trough filled with water is 3m long and its end is a 45° right triangle which is 2m high and 2m wide. Find the work done to pump the water out of the top. (ρ is the density of water and g is the acceleration of gravity.)



a. $2\rho g$
b. $3\rho g$
c. $4\rho g$ correctchoice
d. $6\rho g$
e. $8\rho g$

Put the origin at the bottom of the trough with y positive upward.
A horizontal slice at height y has volume $dV = (\text{length})(\text{width})(\text{height}) = (3)(y)(dy)$ and weight $dF = \rho g\, dV = 3\rho g\, y\, dy$. It must be lifted a distance $D = 2 - y$. So

$$W = \int D\, dF = \int_0^2 (2-y)\, 3\rho g\, y\, dy = 3\rho g \int_0^2 (2y - y^2)\, dy = 3\rho g\left[y^2 - \frac{y^3}{3}\right]_0^2$$

$$= 3\rho g\left[4 - \frac{8}{3}\right] = \rho g[12 - 8] = 4\rho g$$

15. (10 points) Compute $\int_0^1 x\arctan x\ dx$.

Integrate by Parts:
$$u = \arctan x \qquad dv = x\ dx$$
$$du = \frac{1}{1+x^2}\ dx \quad v = \frac{x^2}{2}$$

$$\int_0^1 x\arctan x\ dx = \left[\frac{x^2}{2}\arctan x - \int_0^1 \frac{x^2}{2}\frac{1}{1+x^2}\ dx\right]_0^1 = \left[\frac{x^2}{2}\arctan x - \frac{1}{2}\int_0^1 \frac{x^2+(1-1)}{1+x^2}\ dx\right]$$

$$= \left[\frac{x^2}{2}\arctan x - \frac{1}{2}\int_0^1 1 - \frac{1}{1+x^2}\ dx\right]_0^1 = \left[\frac{x^2}{2}\arctan x - \frac{1}{2}(x - \arctan x)\right]_0^1$$

$$= \left[\frac{1}{2}\arctan 1 - \frac{1}{2}(1 - \arctan 1)\right] = \frac{1}{2}\frac{\pi}{4} - \frac{1}{2}\left(1 - \frac{\pi}{4}\right) = \frac{\pi}{4} - \frac{1}{2}$$

16. (10 points) Solve the differential equation $x^2\frac{dy}{dx} + xy = \frac{2x^2}{1+x^2}$ with the initial condition $y(1) = \ln 4$.

Linear. Put into standard form: $\frac{dy}{dx} + \frac{1}{x}y = \frac{2}{1+x^2} \qquad P = \frac{1}{x} \qquad Q = \frac{2}{1+x^2}$

Integrating factor: $I = e^{\int P\,dx} = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x$

$$\frac{d}{dx}(xy) = x\frac{dy}{dx} + y = \frac{2x}{1+x^2} \qquad xy = \int \frac{2x}{1+x^2}\ dx = \ln(1+x^2) + C$$

Use the initial condition to find C: $x = 1$ when $y = \ln 4$:
$\ln 4 = \ln 2 + C \qquad C = \ln 4 - \ln 2 = \ln 2$

Substitute back and solve for y:
$$xy = \ln(1+x^2) + \ln 2 \qquad y = \frac{\ln(1+x^2)+\ln 2}{x}$$

17. (10 points) Approximate $\int_0^{0.1} e^{-x^2}\,dx$ to 7 decimal places.
(4 points extra credit:) How do you know it is accurate to 7 decimal places?

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots \qquad e^{-x^2} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots$$

$$\int_0^{0.1} e^{-x^2}\,dx \approx \int_0^{0.1} 1 - x^2 + \frac{x^4}{2}\,dx = \left[x - \frac{x^3}{3} + \frac{x^5}{10}\right]_0^{0.1} = .1 - \frac{.001}{3} + .000001 = .099667666\bar{6}$$

Since this is an alternating series, the partial sum is accurate to within the next term which gives
$$\int_0^{0.1}\frac{x^6}{6}\,dx = \left[\frac{x^7}{42}\right]_0^{0.1} = \frac{10^{-7}}{42} \approx 2.4\times 10^{-9} = .0000000024$$
which says the error is in is the 8th digit of the answer.