Fixed Points and Stability, II

MATH 469, Texas A&M University

Spring 2020

Non-dimensionalized Predator-Prey Model

Consider the non-dimensionalized predator-prey model,

$$\begin{aligned} y_{1_{t+1}} - y_{1_t} &= a y_{1_t}(1 - y_{1_t}) - y_{1_t} y_{2_t} \\ y_{2_{t+1}} - y_{2_t} &= -r y_{2_t} + \delta y_{1_t} y_{2_t}, \quad \delta = cK. \end{aligned}$$

We'll identify the fixed points for this system and analyze the stability of each.

The equation for fixed points is

$$\begin{aligned} 0 &= a\hat{y}_1(1 - \hat{y}_1) - \hat{y}_1\hat{y}_2 \\ 0 &= -r\hat{y}_2 + \delta\hat{y}_1\hat{y}_2. \end{aligned}$$

We can write this as

$$\begin{aligned} 0 &= \hat{y}_1(a - a\hat{y}_1 - \hat{y}_2) \\ 0 &= \hat{y}_2(-r + \delta\hat{y}_1). \end{aligned}$$

Fixed Points

From the previous slide:

$$
\begin{aligned}
0 &= \hat{y}_1(a - a\hat{y}_1 - \hat{y}_2) \\
0 &= \hat{y}_2(-r + \delta\hat{y}_1).
\end{aligned}
$$

There are two possibilities for solving the second equation:

$$\hat{y}_1 = \frac{r}{\delta} \quad \text{or} \quad \hat{y}_2 = 0.$$

We can substitute each of these into the first equation, and determine the corresponding value of the other component. We have:

$$
\begin{aligned}
&\hat{y}_1 = \frac{r}{\delta} \implies \frac{r}{\delta}\left(a - a\frac{r}{\delta} - \hat{y}_2\right) = 0 \implies \hat{y}_2 = a\left(1 - \frac{r}{\delta}\right) \\
&\hat{y}_2 = 0 \implies \hat{y}_1(a - a\hat{y}_1) = 0 \implies \hat{y}_1 = 0, 1.
\end{aligned}
$$

Fixed Points

We see that there are three fixed points:

$$\begin{pmatrix} \frac{r}{\delta} \\ a(1-\frac{r}{\delta}) \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We're interested in positive parameter values $a > 0$, $r > 0$, and $\delta > 0$. Also, for the first fixed point, we're primarily interested in the case $\frac{r}{\delta} \leq 1$ (since populations are non-negative). Notice what the fixed points correspond with:

$\begin{pmatrix} \frac{r}{\delta} \\ a(1-\frac{r}{\delta}) \end{pmatrix}$: The two species reach an equilibrium in which neither dies out.

$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$: Both species die out.

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$: The predator species dies out, and the prey reaches its carrying capacity.

The Jacobian Matrix

In order to analyze the stability of these fixed point, we need to construct the Jacobian matrix. If we write our system in the standard form

$$\vec{y}_{t+1} = \vec{f}(\vec{y}_t),$$

we get

$$\begin{aligned} y_{1_{t+1}} &= y_{1_t} + a y_{1_t}(1 - y_{1_t}) - y_{1_t} y_{2_t} \\ y_{2_{t+1}} &= y_{2_t} - r y_{2_t} + \delta y_{1_t} y_{2_t}. \end{aligned}$$

We see that for $\vec{f}(\vec{y}) = \binom{f_1(y_1,y_2)}{f_2(y_1,y_2)}$,

$$\begin{aligned} f_1(y_1, y_2) &= (1 + a)y_1 - a y_1^2 - y_1 y_2 \\ f_2(y_1, y_2) &= y_2 - r y_2 + \delta y_1 y_2. \end{aligned}$$

The Jacobian Matrix

From the previous slide,

$$f_1(y_1, y_2) = (1+a)y_1 - ay_1^2 - y_1 y_2$$
$$f_2(y_1, y_2) = y_2 - ry_2 + \delta y_1 y_2.$$

The partial derivatives we need are as follows:

$$\frac{\partial f_1}{\partial y_1} = 1 + a - 2ay_1 - y_2$$
$$\frac{\partial f_1}{\partial y_2} = -y_1$$
$$\frac{\partial f_2}{\partial y_1} = \delta y_2$$
$$\frac{\partial f_2}{\partial y_2} = 1 - r + \delta y_1.$$

The Fixed Point $\binom{0}{0}$

The Jacobian matrix is

$$\vec{f}'(y_1, y_2) = \begin{pmatrix} 1 + a - 2ay_1 - y_2 & -y_1 \\ \delta y_2 & 1 - r + \delta y_1 \end{pmatrix}.$$

Let's start with $\hat{y} = \binom{0}{0}$. In this case,

$$\vec{f}'(0, 0) = \begin{pmatrix} 1 + a & 0 \\ 0 & 1 - r \end{pmatrix}.$$

The eigenvalues of this matrix are $\lambda_1 = 1 - r$ and $\lambda_2 = 1 + a$. For asymptotic stability, we require *both* of the following conditions to hold:

$$-1 < 1 - r < 1 \implies 1 > r - 1 > -1 \implies 2 > r > 0$$
$$-1 < 1 + a < 1 \implies -2 < a < 0.$$

We see that $\binom{0}{0}$ is always unstable for $a > 0$.

The Fixed Point $\binom{1}{0}$

The Jacobian matrix is

$$\vec{f}'(y_1, y_2) = \begin{pmatrix} 1 + a - 2ay_1 - y_2 & -y_1 \\ \delta y_2 & 1 - r + \delta y_1 \end{pmatrix}.$$

For $\hat{y} = \binom{1}{0}$, we have

$$\vec{f}'(1, 0) = \begin{pmatrix} 1 - a & -1 \\ 0 & 1 - r + \delta \end{pmatrix}.$$

The eigenvalues of this matrix are $\lambda_1 = 1 - r + \delta$ and $\lambda_2 = 1 - a$. For asymptotic stability, we require both of the following conditions to hold:

$$-1 < 1 - r + \delta < 1 \implies 1 > (r - \delta) - 1 > -1 \implies 2 > (r - \delta) > 0$$
$$-1 < 1 - a < 1 \implies 1 > a - 1 > -1 \implies 2 > a > 0.$$

The Fixed Point $\binom{1}{0}$

For $\binom{1}{0}$, we have asymptotic stability if:

$$2 > (r - \delta) > 0 \quad \text{and} \quad 2 > a > 0.$$

Recall that the remaining fixed point is $\binom{\frac{r}{\delta}}{a(1-\frac{r}{\delta})}$. If $\binom{1}{0}$ is asymptotically stable, then $r > \delta$, and the predator population in this remaining fixed point is negative.

Let's check the parameter values we obtained from the hare-lynx data. We found:

$$a = 1.4974, \; b = .0425, \; K = 82.3206, \; r = .5820, \; c = .0239.$$

We can compute $\delta = cK = .0239 * 82.3206 = 1.9675$. In this case

$$r - \delta = -1.3855.$$

This fixed point is unstable for these parameter values.

The Fixed Point $\begin{pmatrix} \frac{r}{\delta} \\ a(1-\frac{r}{\delta}) \end{pmatrix}$

The Jacobian matrix is

$$\vec{f}'(y_1, y_2) = \begin{pmatrix} 1 + a - 2ay_1 - y_2 & -y_1 \\ \delta y_2 & 1 - r + \delta y_1 \end{pmatrix}.$$

In this case,

$$\begin{aligned} \vec{f}'(\frac{r}{\delta}, a(1-\frac{r}{\delta})) &= \begin{pmatrix} 1 + a - 2a\frac{r}{\delta} - a + a\frac{r}{\delta} & -\frac{r}{\delta} \\ \delta a - ar & 1 - r + r \end{pmatrix} \\ &= \begin{pmatrix} 1 - a\frac{r}{\delta} & -\frac{r}{\delta} \\ a(\delta - r) & 1 \end{pmatrix}. \end{aligned}$$

For the eigenvalues of this matrix, we need to compute

$$\det \begin{pmatrix} 1 - a\frac{r}{\delta} - \lambda & -\frac{r}{\delta} \\ a(\delta - r) & 1 - \lambda \end{pmatrix} = 0.$$

The Fixed Point $\begin{pmatrix} \frac{r}{\delta} \\ a(1-\frac{r}{\delta}) \end{pmatrix}$

The characteristic equation is

$$(1 - a\frac{r}{\delta} - \lambda)(1 - \lambda) + \frac{ar}{\delta}(\delta - r) = 0,$$

which we can write as

$$1 - \lambda - a\frac{r}{\delta} + a\frac{r}{\delta}\lambda - \lambda + \lambda^2 + \frac{ar}{\delta}(\delta - r) = 0.$$

Rearraning terms, we obtain

$$\lambda^2 + (\frac{ar}{\delta} - 2)\lambda + (1 - \frac{ar}{\delta} + \frac{ar}{\delta}(\delta - r)) = 0.$$

We can solve this with the quadratic formula:

$$\lambda_{\pm} = \frac{-(\frac{ar}{\delta} - 2) \pm \sqrt{(\frac{ar}{\delta} - 2)^2 - 4(1 - \frac{ar}{\delta} + \frac{ar}{\delta}(\delta - r))}}{2}.$$

The Fixed Point $\begin{pmatrix} \frac{r}{\delta} \\ a(1-\frac{r}{\delta}) \end{pmatrix}$

Let's simplify the discriminant:

$$\begin{aligned}(\frac{ar}{\delta} - 2)^2 \; - 4(1 - \frac{ar}{\delta} + \frac{ar}{\delta}(\delta - r)) \\ = \frac{a^2r^2}{\delta^2} - 4\frac{ar}{\delta} + 4 - 4 + 4\frac{ar}{\delta} - 4\frac{ar}{\delta}(\delta - r) \\ = \frac{a^2r^2}{\delta^2} - 4\frac{ar}{\delta}(\delta - r).\end{aligned}$$

This allows us to write

$$\lambda_{\pm} = 1 - \frac{ar}{2\delta} \pm \frac{1}{2}\sqrt{\frac{a^2r^2}{\delta^2} - 4\frac{ar}{\delta}(\delta - r).}$$

We need to think about two cases:

$$\frac{a^2r^2}{\delta^2} - 4\frac{ar}{\delta}(\delta - r) \geq 0 \quad \text{and} \quad \frac{a^2r^2}{\delta^2} - 4\frac{ar}{\delta}(\delta - r) < 0.$$

The Fixed Point $\begin{pmatrix} \frac{r}{\delta} \\ a(1-\frac{r}{\delta}) \end{pmatrix}$

First, for

$$\frac{a^2 r^2}{\delta^2} - 4\frac{ar}{\delta}(\delta - r) \geq 0,$$

we have

$$\frac{ar}{\delta} - 4(\delta - r) \geq 0 \implies (4 + \frac{a}{\delta})r \geq 4\delta \implies r \geq \frac{4\delta}{4 + \frac{a}{\delta}}.$$

For asymptotic stability, we need

$$-1 < \lambda_-, \lambda_+ < +1.$$

Since $\lambda_- \leq \lambda_+$, we can check two things:

$$-1 < \lambda_- \quad \text{and} \quad \lambda_+ < +1.$$

The Fixed Point $\begin{pmatrix} \frac{r}{\delta} \\ a(1-\frac{r}{\delta}) \end{pmatrix}$

The condition $\lambda_+ < 1$ is

$$1 - \frac{ar}{2\delta} + \frac{1}{2}\sqrt{\frac{a^2r^2}{\delta^2} - 4\frac{ar}{\delta}(\delta - r)} < 1,$$

which we can rearrange as

$$\sqrt{\frac{a^2r^2}{\delta^2} - 4\frac{ar}{\delta}(\delta - r)} < \frac{ar}{\delta} \implies \frac{a^2r^2}{\delta^2} - 4\frac{ar}{\delta}(\delta - r) < \frac{a^2r^2}{\delta^2}.$$

I.e.,

$$-4\frac{ar}{\delta}(\delta - r) < 0 \implies r < \delta.$$

We're already assuming this, so there's nothing new in this case.

The Fixed Point $\left(\begin{smallmatrix} \frac{r}{\delta} \\ a(1-\frac{r}{\delta}) \end{smallmatrix}\right)$

The condition $-1 < \lambda_-$ is

$$-1 < 1 - \frac{ar}{2\delta} - \frac{1}{2}\sqrt{\frac{a^2 r^2}{\delta^2} - 4\frac{ar}{\delta}(\delta - r)},$$

which we can rearrange as

$$\frac{ar}{\delta} - 4 < -\sqrt{\frac{a^2 r^2}{\delta^2} - 4\frac{ar}{\delta}(\delta - r)} \implies 4 - \frac{ar}{\delta} > \sqrt{\frac{a^2 r^2}{\delta^2} - 4\frac{ar}{\delta}(\delta - r)}$$

This is only possible if $4 - \frac{ar}{\delta} > 0$, which we can express as $\delta > \frac{ar}{4}$. In this case, we can square both sides to get

$$16 - 8\frac{ar}{\delta} + \frac{a^2 r^2}{\delta^2} > \frac{a^2 r^2}{\delta^2} - 4ar + 4\frac{ar^2}{\delta}.$$

Rearranging again, we find

$$(4 + ar)\delta > ar(2 + r) \implies \delta > \frac{ar(2 + r)}{4 + ar}.$$

The Fixed Point $\begin{pmatrix} \frac{r}{\delta} \\ a(1-\frac{r}{\delta}) \end{pmatrix}$

We can summarize this as follows: our first criterion for stability is:

$$\begin{aligned} \frac{4\delta}{4+\frac{a}{\delta}} &\leq r < \delta \\ \frac{ar}{4} &< \delta \\ \frac{ar(2+r)}{4+ar} &< \delta. \end{aligned}$$

The Fixed Point $\begin{pmatrix} \frac{r}{\delta} \\ a(1-\frac{r}{\delta}) \end{pmatrix}$

The remaining case is

$$r < \frac{4\delta}{4 + \frac{a}{\delta}},$$

for which $\lambda_\pm$ are complex. In this case

$$\lambda_\pm = 1 - \frac{ar}{2\delta} \pm \frac{1}{2}\sqrt{\frac{a^2r^2}{\delta^2} - 4\frac{ar}{\delta}(\delta - r)}$$
$$= 1 - \frac{ar}{2\delta} \pm \frac{i}{2}\sqrt{-\frac{a^2r^2}{\delta^2} + 4\frac{ar}{\delta}(\delta - r)}.$$

We can compute

$$|\lambda_\pm|^2 = (1 - \frac{ar}{2\delta})^2 + \frac{1}{4}\Big(-\frac{a^2r^2}{\delta^2} + 4\frac{ar}{\delta}(\delta - r)\Big)$$
$$= 1 - \frac{ar}{\delta} + \frac{a^2r^2}{4\delta^2} - \frac{a^2r^2}{4\delta^2} + \frac{ar}{\delta}(\delta - r)$$
$$= 1 + \frac{ar}{\delta}(\delta - r - 1).$$

The Fixed Point $\begin{pmatrix} \frac{r}{\delta} \\ a(1-\frac{r}{\delta}) \end{pmatrix}$

We need

$$1 + \frac{ar}{\delta}(\delta - r - 1) < 1 \implies \frac{ar}{\delta}(\delta - r - 1) < 0 \implies \delta < r + 1.$$

We can summarize this as follows: our second criterion for stability is:

$$\delta - 1 < r < \frac{4\delta}{4 + \frac{a}{\delta}}.$$

We also require $r > 0$, and we're observing that

$$\frac{4\delta}{4 + \frac{a}{\delta}} < \delta.$$

The Fixed Point $\left(\frac{\frac{r}{\delta}}{a(1-\frac{r}{\delta})}\right)$

Again, let's check the parameter values we obtained from the hare-lynx data. We found:

$$a = 1.4974,\ b = .0425,\ K = 82.3206,\ r = .5820,\ c = .0239.$$

We can compute $\delta = cK = .0239 * 82.3206 = 1.9675$.

First, to see which case we're in, we compute

$$\frac{4\delta}{4 + \frac{a}{\delta}} = \frac{4 * 1.9675}{4 + \frac{1.4974}{1.9675}} = 1.6530.$$

Since this value is larger than $r = .5820$, we're in the complex case. Last, we check

$$\delta - 1 = .9675.$$

Since this value of larger than r, we conclude that this fixed point is unstable.

The Fixed Point $\begin{pmatrix} \frac{r}{\delta} \\ a(1-\frac{r}{\delta}) \end{pmatrix}$

Final comment: The instability of this fixed point for our example parameters makes sense, because the solution in that case seemed to be periodic, so we actually expect to find a stable periodic *m*-cycle. We'll consider that next.