Second Order Elliptic PDE: The Lax-Milgram Theorem

MATH 612, Texas A&M University

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The Lax-Milgram Theorem

Let $H$ denote a real Hilbert space with inner product $(\cdot, \cdot)$ and norm $\| \cdot \| = (\cdot, \cdot)^{1/2}$. We’ll continue to use $\langle \cdot, \cdot \rangle$ to denote the action of an element of $H^*$ on an element of $H$.

**Theorem 6.2.1.** (Lax-Milgram Theorem) Suppose $B : H \times H \to \mathbb{R}$ is a bilinear form for which there exist constants $\alpha, \beta > 0$ so that

\[
|B[u, v]| \leq \alpha \|u\| \|v\|, \quad \forall u, v \in H \quad \text{(boundedness)}
\]

\[
B[u, u] \geq \beta \|u\|^2, \quad \forall u \in H \quad \text{(coercivity)}.
\]

Then for each $f \in H^*$, there exists a unique $u \in H$ so that

\[
B[u, v] = \langle f, v \rangle
\]

for all $v \in H$. 
The Lax-Milgram Theorem

**Notes.** 1. If $B$ is symmetric (i.e., $B[u, v] = B[v, u]$ for all $u, v \in H$), then $B[u, v]$ is an inner product on $H$, and this is just the Riesz Representation Theorem.

2. A similar statement is true for a complex Hilbert space $H$, assuming $B$ is sesquilinear. There are also many extensions.

3. Before proving the theorem, we’ll work through a simple application.
Application to Poisson’s Equation

For $U \subset \mathbb{R}^n$ open and bounded, consider Poisson’s equation,

$$-\Delta u = f \in H^{-1}(U), \quad \text{in } U$$
$$u = 0, \quad \text{on } \partial U.$$

The weak formulation for this problem is

$$\int_U \sum_{i,j=1}^n \delta_{ij} u_{x_i} v_{x_j} d\vec{x} = \langle f, v \rangle, \quad \forall v \in H^1_0(U).$$

If we define the bilinear form

$$B[u, v] = \int_U \sum_{i,j=1}^n \delta_{ij} u_{x_i} v_{x_j} d\vec{x} = \int_U Du \cdot Dv d\vec{x},$$

then we can express this weak formulation as

$$B[u, v] = \langle f, v \rangle, \quad \forall v \in H^1_0(U).$$
Application to Poisson’s Equation

We see that our Hilbert space for the Lax-Milgram Theorem is $H^1_0(U)$, and in order to apply the theorem, we only need to check that $B[u, v]$ is bounded and coercive.

For boundedness, we have

$$|B[u, v]| = \left| \int_U Du \cdot Dv dx \right|$$

$$\leq \| Du \|_{L^2(U)} \| Dv \|_{L^2(U)} \leq \| u \|_{H^1(U)} \| v \|_{H^1(U)}.$$

I.e., $\alpha = 1$. 
Application to Poisson’s Equation

For coercivity, we start with

\[ B[u, u] = \int_U |Du|^2 \, d\vec{x} = \|Du\|^2_{L^2(U)}. \]

We recall that Poincare’s inequality (from Theorem 5.6.3) asserts that there exists a constant \( C \), depending only on \( n \) and \( U \), so that

\[ \|u\|_{L^2(U)} \leq C\|Du\|_{L^2(U)}, \]

for all \( u \in H^1_0(U) \).
Application to Poisson’s Equation

We can write

\[ B[u, u] = \frac{1}{2} \| Du \|_{L^2(U)}^2 + \frac{1}{2} \| Du \|_{L^2(U)}^2 \]

\[ \geq \frac{1}{2C^2} \| u \|_{L^2(U)}^2 + \frac{1}{2} \| Du \|_{L^2(U)}^2 \]

\[ \geq \beta \left( \| u \|_{L^2(U)}^2 + \| Du \|_{L^2(U)}^2 \right) \]

\[ = \beta \| u \|_{H^1(U)}^2, \]

for all \( u \in H^1_0(U) \). (Here, \( \beta = \min\{\frac{1}{2C^2}, \frac{1}{2}\} \).)

The Lax-Milgram Theorem allows us to conclude immediately that there exists a unique solution \( u \in H^1_0(U) \) to Poisson’s equation.
1. First, for each fixed $u \in H$, the mapping $T_u v = B[u, v]$ is a bounded linear functional on $H$. I.e.,

$$|T_u v| = |B[u, v]| \leq \alpha \|u\| \|v\|,$$

and linearity follows from the bilinearity of $B$.

We can conclude from the Riesz Representation Theorem that there exists a unique $w \in H$ so that

$$B[u, v] = (w, v), \quad \forall v \in H.$$

Let’s denote by $A : H \to H$ the map that takes $u \in H$ as input and returns $w \in H$ in this way. I.e.,

$$B[u, v] = (Au, v), \quad \forall v \in H.$$
2. **Claim 1.** \( A \in H^* \).

To see that \( A \) is linear, let \( \lambda_1, \lambda_2 \in \mathbb{R}, \ u_1, u_2 \in H \), and compute

\[
\left( A(\lambda_1 u_1 + \lambda_2 u_2), v \right) = B[\lambda_1 u_1 + \lambda_2 u_2, v] \\
= \lambda_1 B[u_1, v] + \lambda_2 B[u_2, v] \\
= \lambda_1 (Au_1, v) + \lambda_2 (Au_2, v) \\
= (\lambda_1 Au_1 + \lambda_2 Au_2, v).
\]

Since this is true for all \( v \in H \), we can conclude that

\[
A(\lambda_1 u_1 + \lambda_2 u_2) = \lambda_1 Au_1 + \lambda_2 Au_2.
\]
Proof of the Lax-Milgram Theorem

To see that $A$ is bounded, we first note that there’s nothing to show if $Au = 0$. If $Au \neq 0$, we compute

$$
\|Au\|^2 = (Au, Au) = B[u, Au] \leq \alpha \|u\| \|Au\|.
$$

Dividing by $\|Au\|$, we see that

$$
\|Au\| \leq \alpha \|u\|.
$$
3. **Claim 2.** $A$ is injective, and the range of $A$, $R(A)$, is closed in $H$.

To see that $A$ is injective, we use coercivity to write

$$\beta \| u \|^2 \leq B[u, u] = (Au, u) \leq \| Au \| \| u \|.$$  

If $\| u \| \neq 0$, we divide to see that

$$\| Au \| \geq \beta \| u \|.$$  

(This is trivially true if $\| u \| = 0$.) In particular, if $u_1, u_2 \in H$, then

$$\| A(u_1 - u_2) \| \geq \beta \| u_1 - u_2 \|,$$  

from which it’s clear that $A$ is injective. (i.e., $u_1 \neq u_2 \implies Au_1 \neq Au_2$.)
Proof of the Lax-Milgram Theorem

To see that \( R(A) \) is closed in \( H \), let \( \{u_j\}_{j=1}^{\infty} \subset H \) satisfy \( Au_j \to w \) for some \( w \in H \). We need to show that there exists \( u \in H \) so that \( Au = w \) (i.e., \( w \in R(A) \)).

For this, we notice that

\[
\|u_i - u_j\| \leq \frac{1}{\beta} \|Au_i - Au_j\|.
\]

The sequence \( \{Au_j\}_{j=1}^{\infty} \) converges, so it must be Cauchy, so we see that \( \{u_j\}_{j=1}^{\infty} \) must be Cauchy, and so must converge to some \( u \in H \). Since \( A \) is bounded,

\[
\|Au - w\| = \lim_{j \to \infty} \|Au - Au_j\| \leq \alpha \lim_{j \to \infty} \|u - u_j\| = 0.
\]

I.e., \( Au = w \). (Alternatively, since \( A \) is bounded, we know from the Closed Graph Theorem that \( A \) is closed, and this allows us to conclude \( Au = w \).)
4. **Claim 3.** $R(A) = H$.

Suppose not, and recall that since $R(A)$ is closed, we can write

$$H = R(A) \oplus R(A)^\perp.$$  

If $R(A) \not\subseteq H$, then we can find $w \in R(A)^\perp \setminus \{0\}$, and for this $w$ we will have

$$0 \neq \beta \|w\|^2 \leq B[w, w] = (Aw, w) = 0,$$

which is a contradiction.

We can conclude from the Bounded Inverse Theorem that $A$ has a bounded linear inverse, $A^{-1}$. 

5. According to the Riesz Representation Theorem, for each \( f \in H^* \) we can find a unique \( w \in H \) so that

\[
\langle f, v \rangle = (w, v)
\]

for all \( v \in H \). In this way, we see that we can solve

\[
B[u, v] = \langle f, v \rangle, \quad \forall v \in H
\]

by solving

\[
B[u, v] = (w, v), \quad \forall v \in H.
\]

Recalling that

\[
B[u, v] = (Au, v), \quad \forall v \in H,
\]

we see that \( Au = w \), and the solution we’re looking for is \( u = A^{-1}w \).
Proof of the Lax-Milgram Theorem

6. For uniqueness, suppose \( u \) and \( \tilde{u} \) are two solutions so that

\[
B[u, v] = \langle f, v \rangle \quad \forall v \in H
\]
\[
B[\tilde{u}, v] = \langle f, v \rangle \quad \forall v \in H.
\]

Subtracting and using linearity, we see that

\[
B[u - \tilde{u}, v] = 0, \quad \forall v \in H.
\]

Take \( v = u - \tilde{u} \), and notice that (from coercivity)

\[
\|u - \tilde{u}\|^2 \leq \frac{1}{\beta} B[u - \tilde{u}, u - \tilde{u}] = 0.
\]

I.e., \( u = \tilde{u} \).