Second Order Elliptic PDE: The Resolvent Operator

MATH 612, Texas A&M University

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Suppose $X$ denotes a real Banach space and $A : X \to X$ denotes a bounded linear operator.

**Definitions.**

(i) The resolvent set of $A$ is defined as

$$\rho(A) := \{ \eta \in \mathbb{R} : (A - \eta I) : X \to X \text{ is bijective} \}.$$  

In particular, for $\eta \in \rho(A)$, $R_A(\eta) := (A - \eta I)^{-1}$ is a bounded linear operator, referred to as the resolvent of $A$ at $\eta$.

(ii) The spectrum of $A$ is defined as

$$\sigma(A) := \mathbb{R} \setminus \rho(A).$$

(Here, we use $\mathbb{R}$ because our Banach space is real.)
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(iii) We denote by $\sigma_p(A)$ the point spectrum (or eigenvalues) of $A$, by which we mean the $\eta \in \mathbb{R}$ so that $N(A - \eta I) \neq \{0\}$. (i.e., $A - \eta I$ is not injective.)

(iv) If $\eta \in \sigma_p(A)$ and $(A - \eta I)w = 0$ for some $w \neq 0$ (i.e., $w \in N(A - \eta I)\{0\}$), then we say $w$ is an eigenvector of $\eta$.

Theorem A.D.6. Suppose $H$ is a real infinite-dimensional Hilbert space, and $K : H \to H$ is compact. Then:

(i) $0 \in \sigma(K)$

(ii) $\sigma(K)\{0\} = \sigma_p(K)\{0\}$

(iii) Either $\sigma(K)$ is finite or $\sigma(K)\{0\}$ is a countable sequence tending toward $0$. 
The Resolvent for Elliptic Operators

**Theorem 6.2.5.** Suppose $U \subset \mathbb{R}^n$ is open and bounded, $a^{ij}$, $b^i$, $c \in L^\infty(U)$ (\(\forall i, j \in \{1, 2, \ldots n\}\)), and $L$ is uniformly elliptic. Then:

(i) There exists a set $\Sigma \subset \mathbb{R}$, at most countable, so that

$$Lu = \lambda u + f, \quad \text{in } U$$

$$u = 0, \quad \text{on } \partial U,$$

has a unique weak solution $u \in H^1_0(U)$ for each $f \in L^2(U)$ if and only if $\lambda \notin \Sigma$.

(ii) If $\Sigma$ from (i) is infinite, then its elements $\Sigma = \{\lambda_k\}^\infty_{k=1}$ can be arranged in a nondecreasing sequence so that

$$\lim_{k \to \infty} \lambda_k = +\infty.$$
Note. Item (i) asserts that for each $\lambda \notin \Sigma$, the operator

$$L - \lambda I : H^1_0(U) \to L^2(U)$$

is a bijection, and correspondingly so is the resolvent operator

$$R_L(\lambda) := (L - \lambda I)^{-1} : L^2(U) \to H^1_0(U).$$
Proof of Theorem 6.2.5

1. Let $\gamma$ be as in Theorem 6.2.2, so that

$$B[u, u] \geq \beta \| u \|_{H^1(U)}^2 - \gamma \| u \|_{L^2(U)}^2, \quad \forall \, u \in H^1_0(U).$$

Notice that if $\gamma$ satisfies this estimate, then any value larger than $\gamma$ also works. Using this, we can take $\gamma > 0$ without loss of generality.

Recall from Theorem 6.2.3 that if $\mu = -\lambda \geq \gamma$, then the equation in Item (i) has a unique weak solution $u \in H^1_0(U)$ for each $f \in L^2(U)$. I.e., we see immediately that if $\lambda \leq -\gamma$, then $\lambda \notin \Sigma$.

We are left to consider $\lambda > -\gamma$. 
Proof of Theorem 6.2.5

2. For any $\lambda > -\gamma$, we’ll apply Theorem 6.2.4 (the Fredholm Alternative) to $L - \lambda I$. According to Theorem 6.2.4, the equation

\[ Lu - \lambda u = f, \quad \text{in } U \]
\[ u = 0, \quad \text{on } \partial U \]

can be uniquely solved for all $f \in L^2(U)$ if and only if $u = 0$ is the only weak solution of

\[ Lu - \lambda u = 0, \quad \text{in } U \]
\[ u = 0, \quad \text{on } \partial U. \]

If we add $\gamma u$ to both sides of this latter equation, we see that it’s equivalent to

\[ Lu + \gamma u = (\gamma + \lambda)u, \quad \text{in } U \]
\[ u = 0, \quad \text{on } \partial U. \]  

(*)
Proof of Theorem 6.2.5

If we now let $K = \gamma L^{-1}_\gamma$ be as in the proof of Theorem 6.2.4, then we can express solutions of (*) as

$$u = L^{-1}_\gamma[\gamma + \lambda]u = \frac{\gamma + \lambda}{\gamma} Ku.$$  

(**)

More precisely, $u \in H^1_0(U)$ solves (*) if and only if it solves (**).

We see that (**) is just an eigenvalue problem for the compact operator $K$,

$$Ku = \frac{\gamma}{\gamma + \lambda} u.$$  

This allows us to apply Theorem A.D.6. It asserts that the allowable values of the ratio $\frac{\gamma}{\gamma + \lambda}$ form a set that is at most countable and tends to 0. Recall that we’re taking $\lambda > -\gamma$, so these allowable values are all positive.
Proof of Theorem 6.2.5

The allowable values of $\frac{\gamma}{\gamma + \lambda}$ can be expressed as

$$\mu_k = \frac{\gamma}{\gamma + \lambda_k},$$

and either the collection $\{\mu_k\}$ is finite or the collection is countable and

$$\lim_{k \to \infty} \mu_k = 0.$$

In the former case, the collection $\Sigma = \{\lambda_k\}$ is finite, while in the latter case it is countable with

$$\lim_{k \to \infty} \lambda_k = +\infty.$$

By a choice of ordering, we can take the set $\{\lambda_k\}$ to be nondecreasing.

\[\square\]
Notes. 1. We saw in Step 2 of the proof that if $\lambda \in \Sigma$, then there exists a solution $u \in H^1_0(U) \backslash \{0\}$ to the equation

$$Lu - \lambda u = 0, \quad \text{in } U$$
$$u = 0, \quad \text{on } \partial U.$$  

I.e., $\Sigma = \sigma_p(L)$.

2. We can proceed similarly in the case that we take $H^1_0(U)$ to be a complex Hilbert space. In that case, Item (i) of Theorem 6.2.5 holds precisely as stated for some (at most) countable set $\Sigma \subset \mathbb{C}$, and Item (ii) holds with

$$\lim_{k \to \infty} |\lambda_k| = +\infty.$$
3. Even if we take $H^1_0(U)$ to be a complex Hilbert space, the eigenvalues of $L$ are confined to $\mathbb{R}$ in many important cases. For example, the eigenvalues associated with the Laplacian operator $L = -\Delta$ are confined to $\mathbb{R}$, and more generally this is the case for operators of the form

$$Lu = - \sum_{i,j=1}^{n} (a^{ij} u_{x_i})_{x_j} + cu,$$

with $a^{ij} = a^{ji}$ for all $i,j \in \{1, 2, \ldots, n\}$. 


Theorem 6.2.6. Suppose $U \subset \mathbb{R}^n$ is open and bounded, $a^{ij}$, $b^i$, $c \in L^\infty(U)$ ($\forall i, j \in \{1, 2, \ldots, n\}$), and $L$ is uniformly elliptic. For $\Sigma$ as in Theorem 6.2.5, if $\lambda \notin \Sigma$ then there exists a constant $C$ so that for any $f \in L^2(U)$ and corresponding (uniquely defined) weak solution $u \in H^1_0(U)$ to

$$Lu = \lambda u + f, \quad \text{in } U$$
$$u = 0, \quad \text{on } \partial U,$$

we have the inequality

$$\|u\|_{L^2(U)} \leq C\|f\|_{L^2(U)}.$$ 

The constant $C$ depends only on $\lambda$, $U$, and the coefficients of $L$. 

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**Note.** In particular, this theorem states that for all $\lambda \notin \Sigma$, the resolvent operator

$$R_L(\lambda) := (L - \lambda I)^{-1},$$

is bounded as a map $R_L(\lambda) : L^2(U) \to L^2(U)$. 
Proof of Theorem 6.2.6

Suppose not. Then for some $\lambda \notin \Sigma$ we can find sequences
\[
\{\tilde{u}_m\}_{m=1}^\infty \subset H_0^1(U) \text{ and } \{\tilde{f}_m\}_{m=1}^\infty \subset L^2(U)
\]
so that
\[
\begin{align*}
L\tilde{u}_m &= \lambda \tilde{u}_m + \tilde{f}_m, \quad \text{in } U \\
\tilde{u}_m &= 0, \quad \text{on } \partial U,
\end{align*}
\]
in the weak sense, but
\[
\|\tilde{u}_m\|_{L^2(U)} > m\|\tilde{f}_m\|_{L^2(U)}
\]
for all $m = 1, 2, \ldots$. 
Proof of Theorem 6.2.6

By a choice of scaling,

\[ u_m := \frac{\tilde{u}_m}{\|\tilde{u}_m\|_{L^2(U)}}, \quad f_m = \frac{\tilde{f}_m}{\|\tilde{u}_m\|_{L^2(U)}}, \]

we can take \( \{u_m\}_{m=1}^{\infty} \) and \( \{f_m\}_{m=1}^{\infty} \) to satisfy

\[
Lu_m = \lambda u_m + f_m, \quad \text{in } U \\
\quad u_m = 0, \quad \text{on } \partial U,
\]

in the weak sense, with \( \|u_m\|_{L^2(U)} = 1 \) and

\[
\|u_m\|_{L^2(U)} > m\|f_m\|_{L^2(U)}
\]

for all \( m = 1, 2, \ldots \).

Since \( \|u_m\|_{L^2(U)} = 1 \) for all \( m = 1, 2, \ldots \), we see that \( \|f_m\|_{L^2(U)} < \frac{1}{m} \to 0 \) as \( m \to \infty \).
Proof of Theorem 6.2.6

Let’s check that the sequence \( \{u_m\}_m^{\infty} \) is bounded in \( H^1_0(U) \).

If \( B[u, v] \) denotes the bilinear form associated with \( L \), then from Theorem 6.2.2 we have the usual lower bound estimate

\[
B[u_m, u_m] \geq \beta \|u_m\|_{H^1(U)}^2 - \gamma \|u_m\|_{L^2(U)}^2, \quad \forall m = 1, 2, \ldots.
\]

This allows us to write

\[
\beta \|u_m\|_{H^1(U)}^2 - \gamma \|u_m\|_{L^2(U)}^2 - \lambda \|u_m\|_{L^2(U)}^2 \leq B[u_m, u_m] - \lambda (u_m, u_m) \\
= (f_m, u_m) \leq \|f_m\|_{L^2(U)} \|u_m\|_{L^2(U)},
\]

and using \( \|u_m\|_{L^2(U)} = 1 \),

\[
\beta \|u_m\|_{H^1(U)}^2 \leq (\gamma + \lambda) + \frac{1}{m}.
\]

We see that \( \{u_m\}_m^{\infty} \) is bounded in \( H^1_0(U) \).
Proof of Theorem 6.2.6

Since $H_0^1(U)$ is reflexive, we can conclude from Theorem A.D.3 that there exists a subsequence $\{u_{m_j}\}_{j=1}^{\infty} \subset \{u_m\}_{m=1}^{\infty}$ so that

$$u_{m_j} \rightharpoonup u \quad \text{in} \quad H_0^1(U).$$

Also, since $H_0^1(U) \subset \subset L^2(U)$, we can extract a further subsequence of $\{u_{m_j}\}_{j=1}^{\infty}$ (though let's continue to use $\{u_{m_j}\}_{j=1}^{\infty}$), so that

$$u_{m_j} \to u \quad \text{in} \quad L^2(U).$$

(By uniqueness of weak limits, and the fact that strong convergence implies weak convergence, the the limits must agree.) Notice particularly that since $\|u_{m_j}\|_{L^2(U)} = 1$ for all $j \in \{1, 2, \ldots\}$, we must have $\|u\|_{L^2(U)} = 1$.

For each $j \in \{1, 2, \ldots\}$, $u_{m_j}$ satisfies the weak problem

$$B[u_{m_j}, v] - \lambda(u_{m_j}, v) = (f_{m_j}, v), \quad \forall \ v \in H_0^1(U).$$
Proof of Theorem 6.2.6

For each \( v \in H^1_0(U) \), the maps

\[
\begin{align*}
  u_{mj} \mapsto B[u_{mj}, v] \\
  u_{mj} \mapsto (u_{mj}, v)
\end{align*}
\]

are bounded linear operators on \( H^1_0(U) \), so by weak convergence we have the limits

\[
\begin{align*}
  \lim_{j \to \infty} B[u_{mj}, v] &= B[u, v] \\
  \lim_{j \to \infty} (u_{mj}, v) &= (u, v).
\end{align*}
\]

In addition,

\[
\lim_{j \to \infty} |(f_{mj}, v)| \leq \lim_{j \to \infty} \|f_{mj}\|_{L^2(U)} \|v\|_{L^2(U)} \leq \lim_{j \to \infty} \frac{\|v\|_{L^2(U)}}{m_j} = 0.
\]
Proof of Theorem 6.2.6

If we take $j \to \infty$ in

$$B[u_{m_j}, v] - \lambda(u_{m_j}, v) = (f_{m_j}, v), \quad \forall v \in H^1_0(U),$$

we obtain

$$B[u, v] - \lambda(u, v) = 0 \quad \forall v \in H^1_0(U).$$

I.e., $u$ is a weak solution of

$$Lu = \lambda u, \quad \text{in } U$$

$$u = 0, \quad \text{on } \partial U.$$ 

Since $\lambda \notin \Sigma$, we know from Theorem 6.2.5 that $u$ is uniquely defined, and since 0 is clearly a solution, this means $u = 0$. But this contradicts our observation that $\|u\|_{L^2(U)} = 1$, and so contradicts our original assumption that there exists a value $\lambda \notin \Sigma$ for which the claimed estimate doesn't hold. \qed