Ill-posedness for the 2-Dimensional Euler equations

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The incompressible Euler equation in $\mathbb{R}^2$

\[
\begin{aligned}
  u_t + u \cdot \nabla u &= -\nabla p \quad \text{(balance of momentum)} \\
  \text{div } u &\equiv 0 \quad \text{(incompressibility condition)}
\end{aligned}
\]

$u =$ \text{fluid velocity} \quad p =$ \text{pressure}

\[
\begin{aligned}
  u_t + u \cdot \nabla u &= D_t u = \text{material derivative} \\
                          &\quad = \text{acceleration of a fluid particle}
\end{aligned}
\]
Smooth solutions of the incompressible Euler equation are unique and conserve energy.

Onsager

- Energy is not conserved (Red)
- Non-uniqueness

Yudovich

- Energy is conserved (Blue)
- Uniqueness

$C^{1/3}$
Uniqueness for the incompressible Euler equations

- \( \text{curl } u \in L^\infty(\mathbb{R}^2) \implies \text{solution is unique} \)


**Non-uniqueness:**


- Yields infinitely many solutions to the Cauchy problem, by a Baire category argument, all with low regularity, none can be explicitly described
- One may still hope that, by imposing further admissibility conditions, a “good solution” can be selected, depending continuously on initial data
- ... but no such condition has ever been found
Non-uniqueness for ODEs

**Example 1:**  
\[
\dot{x} = |x|^{1/2} \quad x(0) = \bar{x}
\]

A solution can be selected, depending continuously on the initial datum: namely, the **unique strictly increasing solution**

**Example 2:**  
\[
\dot{x} = x^{1/3} \quad x(0) = \bar{x}
\]

There is no way to select a solution depending continuously on the initial datum
Is there a simple “textbook” example showing that

\[ \text{curl } u \in L^p(\mathbb{R}^2) \setminus L^\infty(\mathbb{R}^2) \implies \text{Cauchy problem is ill-posed} \]
Vorticity formulation of the Euler equations

\[
\begin{aligned}
\begin{cases}
  u_t + u \cdot \nabla u &= -\nabla p \\
  \text{div } u &\equiv 0
\end{cases}
\end{aligned}
\]

\[u = \text{velocity}, \quad \omega = \text{vorticity}, \quad \psi = \text{stream function}\]

\[\omega = \text{curl } u = -u_{1,x_2} + u_{2,x_1}\]

\[u = \nabla^\perp \psi, \quad (u_1, u_2) = (-\psi_{x_2}, \psi_{x_1})\]

\[
\begin{aligned}
\begin{cases}
  \omega_t + \nabla^\perp \psi \cdot \nabla \omega &= 0 \\
  \Delta \psi &= \omega
\end{cases}
\end{aligned}
\]
Self-similar solutions

Scaling parameter: $\mu > \frac{1}{2}$, $y = \frac{x}{t^\mu}$

$$
\begin{align*}
  u(t, x) &= t^{\mu-1} U \left( \frac{x}{t^\mu} \right) \quad \text{(velocity)} \\
  \omega(t, x) &= t^{-1} \Omega \left( \frac{x}{t^\mu} \right) \quad \text{(vorticity)} \\
  \psi(t, x) &= t^{2\mu-1} \psi \left( \frac{x}{t^\mu} \right) \quad \text{(stream function)}
\end{align*}
$$

$t = 1 \implies (u, \omega, \psi) = (U, \Omega, \Psi)$

$$
\begin{align*}
  \left( \nabla^{\perp} \psi - \mu y \right) \cdot \nabla \Omega &= \Omega \\
  \Delta \psi &= \Omega
\end{align*}
$$

velocity: $U = \nabla^{\perp} \psi$
Constructing two solutions with the same initial data

initial vorticity: $\bar{\omega}(x) \approx r^{-1/\mu} \cdot \phi(\theta)$

(support of the initial vorticity)
\[ \bar{\omega}(x) \approx r^{-1.4} \phi(\theta) \]

setting \( \bar{\omega}(x) = \varepsilon^{-1/\mu} \) for \( |x| < \varepsilon \)
\[ \bar{\omega}(x) \approx r^{-1.4} \phi(\theta) \]

setting \( \bar{\omega}(x) = 0 \) for \(|x| < \varepsilon\)
Plots with vorticity $\overline{\omega} \in L^\infty(\mathbb{R}^2)$, $\overline{\omega}(x) = \phi(\theta)$

setting $\overline{\omega}(x) = 1$ for $|x| < \varepsilon$

setting $\overline{\omega}(x) = 0$ for $|x| < \varepsilon$
Changing parameter values

\[ \overline{\omega}(x) \approx r^{-1/\mu} \cdot \phi(\theta) \]

- If \( \phi \) is supported on a large angle \( \Theta \), one always finds a single spiral.
- If the exponent \( \alpha = \frac{1}{\mu} \) is small, one always obtains two spirals.
More simulations  
(Wen Shen, 2019)  
\[ \alpha = 0.95, \quad \Theta = \frac{2\pi}{3} \]
$\alpha = 0.95, \quad \Theta = \pi/4$
Goal: validate these numerical simulations by rigorous a posteriori error estimates

Main difficulties:

- Solution is defined on the unbounded domain $\mathbb{R}^2$
- Lack of regularity near the spirals’ centers

Need: a domain decomposition method (A.B. - R.Murray, 2019)
On the outer domain $\mathcal{D}^\#$, the solution can be constructed analytically, given the asymptotic conditions as $|x| \to +\infty$

On the inner domain $\mathcal{D}^\flat$, the solution has a spiraling singularity. It can be constructed analytically, using an adapted system of coordinates (V.Elling)

On the intermediate domain $\mathcal{D}^\natural$, the solution can be numerically computed. Since here the vorticity $\Omega$ is smooth, a posteriori error estimates yield the existence of an exact solution, close to the numerical approximation

The three solutions should be connected by suitable matching conditions
Solutions on the outer domain

\[ \mathcal{D}_R \doteq \{ x \in \mathbb{R}^2 ; \ |x| > R \} \]

Find a solution to

\[
\begin{cases}
(\nabla^\perp \psi - \mu x) \cdot \nabla \Omega = \Omega \\
\Delta \psi = \Omega
\end{cases}
\]

satisfying the boundary and asymptotic conditions

\[
\begin{cases}
\psi(R, \theta) = \psi_0(\theta) \\
\lim_{r \to +\infty} r^{\frac{1}{\mu}} \Omega(r, \theta) = \bar{\omega}(\theta) \\
\lim_{r \to +\infty} r^{\frac{1}{\mu} - 2} \psi(r, \theta) = \bar{\psi}(\theta)
\end{cases}
\]

- The functions \( \bar{\omega}, \bar{\psi} : [0, 2\pi] \mapsto \mathbb{R} \) must satisfy the compatibility condition

\[
\bar{\psi}_{\theta\theta} + \left(2 - \frac{1}{\mu}\right)^2 \bar{\psi}(\theta) = \bar{\omega}(\theta)
\]

- Solving (*) on \( \mathcal{D}_R \) is the same as solving (*) on \( \mathcal{D}_1 \), with smaller data:

\( (\psi_0, \bar{\omega}, \bar{\psi}) \) is replaced by \( (R^{-2} \psi_0(\theta), R^{-\frac{1}{\mu}} \bar{\omega}(\theta), R^{-\frac{1}{\mu}} \bar{\psi}(\theta)) \)
A sequence of approximations \( D_1 = \{ x \in \mathbb{R}^2; |x| > 1 \} \)

\[
\begin{cases}
(\nabla \perp \psi_{n-1} - \mu x) \cdot \nabla \Omega_n &= \Omega_n \\
\Delta \psi_n &= \Omega_n
\end{cases}
\]

\[
\psi_n(x) = \psi_{n-1}(x) + \frac{1}{2\pi} \int_{|y|>1} \ln \left( \frac{1}{|y|} \cdot \frac{|y-x|}{|y^* - x|} \right) (\Omega_n(y) - \Omega_{n-1}(y)) \, dy
\]

To achieve integrability one needs: \( \Omega_n(y) - \Omega_{n-1}(y) = \mathcal{O}(1) \cdot |x|^{-2-\frac{1}{\mu}} \)

\( \Rightarrow \) initial guess \((\Omega_0, \psi_0)\) should be sufficiently accurate
Solutions on the intermediate domain \( D^\# \)

By a numerical simulation, find a smooth solution to

\[
\begin{cases}
(\nabla^\perp \Psi - \mu x) \cdot \nabla \Omega = \Omega \\
\Delta \Psi = \Omega
\end{cases}
\]

on an intermediate domain of the form

- Stream function \( \Psi \) is assigned on the entire boundary \( \partial D^\# \)
- Vorticity \( \Omega \) is assigned on the outer boundary \( \Sigma_1 \)
Solutions on an inner domain \( \{ y \in \mathbb{R}^2 ; \ |y| < R \} \)

\[
\begin{aligned}
\left\{ \begin{array}{ll}
(\nabla \perp \psi - \mu y) \cdot \nabla \Omega &= \Omega \\
\Delta \psi &= \Omega
\end{array} \right.
\end{aligned}
\]

- \( q(y) = \nabla \perp \psi(y) - \mu y \) is the **pseudo-velocity**
- the integral curves of \( q \) are the **pseudo-streamlines**

To resolve the singularity at the spiral center, one needs to use a set of adapted coordinates, following the pseudo-streamlines (Volker Elling, 2013, 2016)
A system of adapted coordinates on the disc \( \{ y \in \mathbb{R}^2, |y| < R \} \)

Variable change: \((y_1, y_2) \mapsto (\beta, \phi)\)

- Pseudo-streamlines have equation \( \phi = \text{constant} \).
- \( \theta = \phi + \beta \), where \( r = |y|, \theta = \angle y \) are the polar coordinates of \( y \).
- For fixed \( \phi \) we have

\[
\begin{align*}
  r(0, \phi) &= R, & \theta(0, \phi) &= \phi, & \lim_{\beta \to +\infty} r(\beta, \phi) &= 0
\end{align*}
\]
The stream function $\Psi$ in adapted coordinates (V. Elling, 2013)

\[
\left( \nabla \psi - \mu y \right) \cdot \nabla (\Delta \psi) = \Delta \psi
\]

\[
\left[ \left( 1 + \left( \frac{\psi_{\beta \phi}}{2\psi_{\beta}} \right)^2 \right) \frac{2\psi_{\beta} \psi_{\phi}}{\psi_{\beta \phi}} - \frac{\psi_{\beta \phi} \psi_{\phi}}{2\psi_{\beta}} \right] \phi + \left[ \frac{\psi_{\beta \phi} \psi_{\phi} - \psi_{\beta \phi} \psi_{\phi}}{2\psi_{\beta}} \right] \phi = -\frac{\psi_{\beta \phi}}{2\mu} \psi_{\phi} \frac{1}{2\mu} \Omega(\phi)
\]

defined for $\beta > 0$, $\phi \in [0, 2\pi]$.

Here $\partial_\phi \equiv \partial_\phi - \partial_\beta$

boundary condition: $\Psi(0, \phi) = \overline{\Psi}(\phi)$.
Radially symmetric solutions: $\Psi = \Psi(r)$

\[
\begin{align*}
\left\{ \begin{array}{l}
(\nabla \perp \psi - \mu y) \cdot \nabla \Omega &= \Omega \\
\Delta \psi &= \Omega 
\end{array} \right. \\
\left\{ \begin{array}{l}
-\mu r \Omega_r &= \Omega \\
\psi_{rr} + \frac{\psi_r}{r} &= \Omega 
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
    r &= \kappa \beta^{-\mu}, \\
    \Omega &= \Omega_0 \beta, \\
    \psi &= \frac{\mu \kappa^2}{2\mu-1} \beta^{1-2\mu},
\end{cases}
\end{align*}
\]

The algebraic spiral $\theta = \frac{3}{2} r^{-4/3}$ obtained by taking $\kappa = 1, \mu = 3/4$. 
Linearizing around the radially symmetric solution

- Meaningful, because near the spiral center the solution is “almost” radially symmetric

\[ \Psi_\epsilon = \Psi_0 + \epsilon Y(\phi, \beta) + o(\epsilon) \]

\[
\left( \beta \partial_\phi + (2\mu - 1) + i \mu \partial_\phi \right) \left( \beta \partial_\phi + (2\mu - 1) - i \mu \partial_\phi \right) (\beta \partial_\beta + 1) Y + (2\mu - 1) \beta Y_\phi = \tilde{\omega}(\phi)
\]

- third order, non-homogeneous linear equation
- can be studied by Fourier transform
Existence of self-similar spiraling solutions

\[\begin{aligned}
\left\{ \begin{align*}
\left( \nabla^\perp \Psi - \mu y \right) \cdot \nabla \Omega &= \Omega \\
\Delta \Psi &= \Omega
\end{align*} \right. \\
\end{aligned}\]

\[
\left[ \left( 1 + \left( \frac{\Psi_{\beta \phi}}{2\Psi_{\beta}} \right)^2 \right) \frac{2\Psi_{\beta} \Psi_{\phi}}{\Psi_{\beta \phi}} - \frac{\Psi_{\beta \phi} \Psi_{\phi}}{2\Psi_{\beta}} \right]_{\phi} + \left[ \frac{\Psi_{\beta \phi} \Psi_{\phi} - \Psi_{\beta \phi} \Psi_{\phi}}{2\Psi_{\beta}} \right]_{\phi} = -\frac{\Psi_{\beta \phi}}{2\mu} \Psi_{\phi}^{-\frac{1}{2\mu}} \Omega(\phi)
\]

by a perturbation argument, one can construct solutions
for data which are sufficiently close to radially symmetric

V. Elling, Algebraic spiral solutions of 2d incompressible Euler.

V. Elling, Self-Similar 2d Euler solutions with mixed-sign vorticity.
Near the spiral center, the solution is close to radially symmetric. Local solutions can be constructed by using adapted coordinates (R. Murray, 2019).

These can be patched together with numerically computed solutions in an intermediate domain, and analytic solutions in an outer domain.
Matching procedure

\[
\begin{align*}
\left\{ \begin{array}{l}
(\nabla^\perp \psi - \mu x) \cdot \nabla \Omega = \Omega \\
\Delta \psi = \Omega
\end{array} \right. \\
\Omega_1 = \Omega_2, \quad \psi_1 = \psi_2, \quad \partial_n \psi_1 = \partial_n \psi_2 \quad \text{for all } x \in \Sigma_2
\end{align*}
\]

Assume: solutions \((\Omega_1, \psi_1)\) and \((\Omega_2, \psi_2)\) are constructed separately on

\[
D_1 = \{ r_1 \leq |x| \leq r_2 \}, \quad D_2 = \{ |x| > r_2 \}
\]

Need: matching conditions on the boundary \(\Sigma_2 = \{|x| = r_2\}\)

\[
\begin{align*}
\lim_{r \to +\infty} r^{\frac{1}{\mu}} \frac{1}{\Omega(r, \theta)} &= \bar{\omega}(\theta) \\
\lim_{r \to +\infty} r^{\frac{1}{\mu} - 2} \psi(r, \theta) &= \bar{\psi}(\theta)
\end{align*}
\]

Alberto Bressan (Penn State)
Overlapping domains

More convenient: choose \( r'_2 < r_2 \) and construct solutions \( (\Omega_1, \Psi_1) \) and \( (\Omega_2, \Psi_2) \) on the overlapping domains

\[
D_1 = \{ r_1 \leq |x| \leq r_2 \}, \quad D_2 = \{ |x| > r'_2 \}
\]

Then require

\[
\Omega_1 = \Omega_2, \quad \Psi_1 = \Psi_2, \quad x \in \Sigma_2 = \{|x| = r_2\}
\]

\[
\Psi_1 = \Psi_2, \quad x \in \Sigma'_2 = \{|x| = r'_2\}
\]

(can be achieved by a fixed point argument)
Summary of results

On the **outer domain** \( \{ x \in \mathbb{R}^2 ; |x| > 1 \} \), for \( \mu > 0, \mu, 2\mu \notin \mathbb{N} \), a unique solution exists for all asymptotic and boundary data \( \bar{\omega}, \bar{\psi}, \psi_0 \) sufficiently small, satisfying the compatibility condition (A.B., 2019)

On the **inner domain**, for \( \mu > 2 \), a unique solution exists for boundary data \( \omega_1, \psi_1 \) sufficiently close to radially symmetric (R.Murray, 2019)

The existence of a solution defined on the entire plane \( \mathbb{R}^2 \) is thus reduced to a problem that can be decided by a numerical computation on an intermediate domain, in finite precision.

**A posteriori error estimates**: work in progress (A.B., W.Shen, L.Zikatanov)

The analysis covers solutions with \( \text{curl} \ u(x) = O(1) \cdot |x|^{-1/\mu} \in L^p_{loc}(\mathbb{R}^2) \)
what about compressible flow?
Compressible, inviscid fluid flow

Hyperbolic systems of conservation laws

- In one space dimension, the Cauchy problem is well posed in $L^1$ as long as the total variation remains bounded.

- In two or more space dimensions the Cauchy problem is hopelessly ill posed?
\[ \frac{d}{dt} u = \Phi(u) \quad u(0) = \bar{u} \in X \quad (CP) \]

**Definition.**

The Cauchy problem \((CP)\) is **incurably ill posed** on the space \(X\) if there exists dense subsets \(\mathcal{D}^u, \mathcal{D}^m \subset X\) such that the following holds.

- For each initial datum \(\bar{v} \in \mathcal{D}^u\), there exists a time \(T(\bar{v}) > 0\) such that \((CP)\) has a unique local solution \(v(t) = S_t \bar{v}, \quad t \in [0, T(\bar{v})]\).

- For each initial datum \(\bar{u} \in \mathcal{D}^m\), there exist two sequences \(\bar{v}_n \to \bar{u}, \bar{w}_n \to \bar{u}\), with \(\bar{v}_n, \bar{w}_n \in \mathcal{D}^u\), such that for some \(\tau > 0\) the corresponding sequences of solutions converge to different limits:

\[ v_n(t) \to u(t), \quad w_n(t) \to \tilde{u}(t), \quad u(t) \neq \tilde{u}(t) \quad \text{for all } t \in ]0, \tau] \]

\[ \Rightarrow \quad \text{the map } (t, \bar{u}) \mapsto S_t \bar{u}, \quad \bar{u} \in \mathcal{D}^u, \quad \text{does NOT admit any continuous extension to any open set} \]
Incurably ill posed problems

Is this what happens for multi-dimensional hyperbolic conservation laws?
The p-system of isentropic gas dynamics

\[
\begin{align*}
\partial_t \rho + \text{div} (\rho v) &= 0 & \text{conservation of mass} \\
\partial_t (\rho v) + \text{div} (\rho v \otimes v) + \nabla(p(\rho)) &= 0 & \text{conservation of momentum}
\end{align*}
\]

\[\rho = \text{density} \quad v = \text{velocity} \quad p(\rho) = \text{pressure}\]

Conjecture: in two space dimensions this system is \textit{incurably ill-posed} in any space containing $H^1$.

- every initial datum can be approximated with a smooth one
- every smooth initial datum admits small $H^1$ perturbations, with vorticity $\omega \in L^2(\mathbb{R}^2)$ supported on two wedges, leading to two distinct solutions
A key technical issue

- V. Elling’s method of adapted coordinates applies to self-similar solutions which are close to radially symmetric on the entire plane.

- R. Murray’s work extends this method to self-similar solutions which are close to radially symmetric only near the spiral’s center.
However, an arbitrarily small $C^\infty_c$ perturbation of initial data destroys the self-similarity. All the above techniques break down.

Solutions to the incompressible Euler equation on a bounded domain are not self-similar.

Solutions to the slightly compressible Euler equations do not have the appropriate self-similarity properties.

We should learn how to use adapted coordinates to construct solutions $u = u(x_1, x_2, t)$ which are only “asymptotically self-similar” near a spiral center.
Happy Birthday Kevin!