Spiral waves: linear and nonlinear theory

Björn Sandstede

Margaret Beck

Stephanie Dodson

Toan Nguyen

Arnd Scheel

Kevin Zumbrun
Spiral waves

cAMP signalling of amoebae [Newell]

CO oxidation on platinum [Nettesheim, von Oertzen, Rotermund, Ertl]

Belousov-Zhabotinsky reaction [Swinney et al.]

Calcium waves in Xenopus oocytes [Clapham et al.]
Spiral waves

Dynamics of core / spiral tip
Modulations of wave trains in far field

[Li, Ouyang, Petrov, Swinney]
Spiral waves in cardiac tissue

- Cardiac arrhythmias can be caused by spiral waves pinned to inhomogeneities
- Low-energy excitation can remove spiral waves [Luther et al.]
- Proposed strategies are sensitive to the relative phase difference between excitation waves and externally induced stimuli
Spiral waves in cardiac tissue

Ventricular Tachycardia (VT)
Abnormally fast heart rate

Ventricular Fibrillation (VF)
Fatal unless treated immediately

Alternans rhythm

Arrhythmia pattern
Spatio-temporal period-doubling of spiral waves

Alternans in cardiac tissue (subcritical?)

Line defects in light-sensitive BZ-reaction [Yoneyama, Fujii, Maeda] (supercritical?)
Outline

- Wave trains: spectral stability and modulation equations
- Planar spiral waves: spiral spectra on the plane and on bounded disks
- One-dimensional defects: anticipated dynamics, and nonlinear stability results
Slowly varying modulations

\[ u(x, t) = u_*(kx - \omega_*(k)t; k) \]

Harmonic modulations

\[ u_*(kx - \omega_*(k)t + \epsilon e^{\lambda t} \cos(\gamma x)) \]
\[ \approx u_*(kx - \omega_*(k)t) + \epsilon e^{\lambda t} \cos(\gamma x)u'_w t(kx - \omega_*(k)t) \]

Linear temporal response to spatial modulation with wavenumber \( \gamma \) is given by \( \lambda = \lambda_*(i\gamma) \)

Spectrum of wave trains

\[ \lambda_*(i\gamma) = -ic_g \gamma - d\gamma^2 + O(|\gamma|^3) \]

\[ e^{\lambda t} \cos(\gamma x) \approx e^{-ic_g \gamma t} e^{i\gamma x} \]
\[ = e^{i\gamma(x - c_g t)} = \cos(\gamma(x - c_g t)) \]
Slowly varying modulations

Viscous Burgers equation: \[ q_T = \lambda^\prime\prime(0)q_{XX} - \omega^\prime\prime(k)(q^2)_X \]
for slowly varying wavenumber modulations \(q(X,T)\)
on scale \(X=\varepsilon(x-c_gt)\) and \(T=\varepsilon^2t/2\) with \(0<\varepsilon\ll 1\)

- Formal derivation: [Howard & Kopell], [Kuramoto]
- Validity over natural time scale \(1/\varepsilon^2\): [Doelman, S., Scheel, Schneider]
- Stability of wave trains: [S., Scheel, Schneider, Uecker], [Johnson, Zumbrun], [Iyer, S.]
Outline

- Wave trains: spectral stability and modulation equations
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Planar spiral waves

Reaction-diffusion systems: \[ u_t = D\Delta u + f(u) \]

Rotating spiral waves: \[ u(x, t) = u_*(r, \phi - \omega_* t) \]

Linearization in co-rotating frame: \[ \mathcal{L}_* v = D\Delta v + f_u(u_*) v + \omega_* v_\phi \]

Archimedean spiral waves: \[ u_*(r, \phi) \approx u_{wt}(\kappa r + \phi), \quad r \gg 1 \]
\[ \mathcal{L}_* \approx D\partial_{rr} + f_u(u_{wt}(\kappa r + \phi)) + \omega_* \partial_\phi \]

**Theorem [S., Scheel]** Assume that \( u_*(r, \phi) \) is a generic spiral wave so that the asymptotic wave train has \( c_g > 0 \), then there is an \( a \neq 0 \) such that \( |u_*(r, \phi) - u_{wt}(\kappa r + \phi - a \log r)| \to 0 \) as \( r \to \infty \).

**Goals:**
- Understand the structure of the spectrum of the linearization about a spiral wave
- Relate the spectra of asymptotic wave train and spiral wave
Spectra of planar spiral waves

Linear dispersion relation of asymptotic wave train in laboratory frame: $\lambda(i\gamma)$

Far field eigenfunctions: $v(r, \phi) \approx e^{i\gamma r} e^{i\ell \phi} \nu_\infty(\kappa r + \phi)$
Spectra of planar spiral waves

Spectrum on spaces with exponentially decaying weights $e^{-r}$

- Adjoint eigenfunctions associated with eigenvalues $\lambda = 0, \pm i\omega_*$ are exponentially localized
- $\Sigma_{\text{ext}}(\text{core}) :=$ point spectrum in weighted spaces
Spiral waves on large disks

Planar spiral wave

Boundary sink

Spiral wave on large disk:
- Glue planar spiral wave and boundary sink together
- Time (sink) = angle (spiral)
Spectra of spiral waves on disks of radius $R$

Absolute spectrum: depends on wave train only ($1/R$ convergence)
Discrete eigenvalues: depend on spiral wave ($\exp(-R)$ convergence)

Theorem [S., Scheel]

$$\Sigma(\text{bounded disk}) \to \Sigma_{\text{abs}} \cup \Sigma_{\text{ext}}(\text{core}) \cup \Sigma_{\text{ext}}(\text{boundary sink}) \text{ as } R \to \infty.$$
Case study: Period doubling of spiral waves

Line defects in Rössler model

Alternans in Karma model

Goal:
- What causes these instabilities?
- Core, boundary, or absolute spectrum?
Case study: Period doubling of spiral waves

Line defects in Rössler model

Alternans in Karma model

\begin{align*}
\lambda &\approx 3i\omega/2 \\
\lambda &\approx i\omega/2
\end{align*}

\begin{align*}
\lambda &\approx 3i\omega/2 \\
\lambda &\approx i\omega/2
\end{align*}
Case study: Period doubling of spiral waves

Methodology: Compute spectra separately for
- Spiral wave on disk with Neumann boundary conditions: $u_r = 0$ at $r=R$
- Spiral wave on disk with non-reflecting boundary conditions: $u_r = \kappa u_\phi$ at $r=R$
- Boundary sink on $(-L,0]$ with Neumann conditions at $x = 0$

![Neumann spiral](image1)
- Absolute spectrum
- Core spectrum
- Boundary spectrum

![Non-reflecting spiral](image2)
- Absolute spectrum
- Core spectrum

![Boundary sink](image3)
- Absolute spectrum
- Core spectrum
- Boundary spectrum
Case study: Period doubling of spiral waves

Line defects in Rössler model

Alternans in Karma model

Result:
- Line defect structure caused by Neumann boundary conditions
- Alternans caused by spiral core: should arise independently of the boundary conditions
Zero-diffusion limit of spiral spectra

\[ u_t = \Delta u + f(u, v) \]
\[ v_t = \delta \Delta v + g(u, v) \]
Zero-diffusion limit of spiral spectra

\[ u_t = \Delta u + f(u, v) \]
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Far field eigenfunctions: \[ v(r, \phi) \approx e^{i\gamma r} e^{i\ell \phi} v_\infty (\kappa r + \phi) \]
Zero-diffusion limit of spiral spectra

\[ u_t = \Delta u + f(u, v) \]
\[ v_t = \delta \Delta v + g(u, v) \]

**Theorem [Dodson, S.]:**
Assume that \( g(u, v) = h(u) + av \) and that \( u_*(r, \phi; \delta) \) is a nondegenerate planar spiral wave that depends smoothly on \( \delta \), then the essential spectrum of the linearization about the spiral wave is discontinuous in the limit \( \delta \to 0 \) near the points \( \lambda = a + i\ell \).

**[Rademacher]:** Wave-train spectrum is continuous in the limit \( \delta \to 0 \)
Outline

- Wave trains: spectral stability and modulation equations
- Planar spiral waves: spiral spectra on the plane and on bounded disks
- Nonlinear stability of one-dimensional spiral waves
Sources

\[ \frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + f(u), \quad x \in \mathbb{R}, \quad u \in \mathbb{R}^n \]

\( u_*(x, t) \) is periodic in time (possibly in moving frame) and spatially asymptotic to two (possibly different) wave trains

Standing time-periodic structure
Spectra of sources

\[ \frac{\partial \nu}{\partial t} = D \frac{\partial^2 \nu}{\partial x^2} + f_u(u_*(x, t)) \nu \]

Spectrum of wave trains

\[ L^2 \text{ space} \]

\[ L^2_{\text{exp}} := \{ \nu : \| \nu(x) e^{-|x|} \|_{L^2} < \infty \} \]
Nonlinear stability of sources

- Defect core converges exponentially to new position
- Modulation interface is localized and travels with speed given by the group velocity
Nonlinear stability of sources

Unperturbed source

Difference of perturbed and unperturbed source
Nonlinear stability of sources

Theorem [Beck, Nguyen, S., Zumbrun]:
Assume $u_*(x, t)$ is a spectrally stable source, and let $u(x,0) = u_*(x,0) + v_0(x)$ with $v_0(x)$ small in a weighted norm. Then there are small constants $p_\infty$ and $\varphi_\infty$ such that

\[
|u(x, t) - u_*(x - p_\infty, t - \varphi_\infty)| < \varepsilon Ce^{-\eta t} \text{ in } \Omega_1
\]

\[
|u(x, t) - u_*| < \varepsilon Ce^{-\eta t} \text{ in } \Omega_2
\]
Summary and outlook

Summary:
- stability of wave trains under classes of localized perturbations
- robustness and structure of planar spiral waves and target patterns
- structure of spectra of spiral waves on the plane and on bounded disks
- nonlinear stability of 1d sources

Open problems:
- stability of wave trains under perturbations of the wave number
- linear stability of planar spiral waves
- interaction of spirals (or other waves)