Spectral Stability of Transition Front Solutions in Multidimensional Cahn-Hilliard Systems

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SIAM Conference on Nonlinear Waves and Coherent Structures,
Aug. 11-14, 2014

To appear in Journal of Differential Equations
Outline of the Talk

▶ Introduction to Cahn-Hilliard Systems
▶ Transition Fronts
▶ Structure of the Linearized Equation
▶ Main Spectral Theorem
▶ Remark on Nonlinear Stability
▶ Elements of the Proof (time permitting)
▶ Theorem on Nonlinear stability (time permitting)
Cahn-Hilliard Systems

For \( x \in \mathbb{R}^n \) and \( u \in \mathbb{R}^m \), we consider systems of the form

\[
\frac{\partial u_j}{\partial t} = \nabla \cdot \left\{ \sum_{k=1}^{m} M_{jk}(u) \nabla \left( (-\Gamma \Delta u)_k + F_{uk}(u) \right) \right\},
\]

or in tensor notation

\[
u_t = \nabla \cdot \left\{ M(u)D_x \left( -\Gamma \Delta u + D_u F \right) \right\}.
\]

If \( M \) is the identity matrix,

\[
u_t = \Delta (-\Gamma \Delta u + D_u F).
\]

This is a standard model of certain phase separation processes such as spinodal decomposition, where the components of \( u \) characterize \( m \) components of a mixture that contains \( m + 1 \) components in all.
Cahn-Hilliard Systems

Our equation is

\[ u_t = \nabla \cdot \left\{ M(u) D_x \left( -\Gamma \Delta u + D_u F \right) \right\}. \]

Here, \( F \in \mathbb{R} \) is a measure of bulk free energy density, \( M \in \mathbb{R}^{m \times m} \) is a measure of molecular mobility, and \( \Gamma \in \mathbb{R}^{m \times m} \) characterizes interfacial energy. Based on physical considerations, we assume \( M \) and \( \Gamma \) are symmetric and positive definite, \( M \) uniformly so. In fact, for the main result of this talk we’ll take \( M \) constant.

We are interested in the spectrum of the linear operator obtained when this system is linearized about a transition front \( \bar{u}(x_1) \).
Historical Remark on Terminology


Cahn-Hilliard systems were first studied by Didier de Fontaine in his 1967 Northwestern thesis “A computer simulation of the evolution of coherent composition variations in solid solutions,” carried out under the direction of John Hilliard.

de Fontaine explains that Hilliard was never comfortable having his name on the equation and referred to it himself as “the last unnumbered equation after equation (18) in Cahn’s 1961 paper.”
The Bulk Free Energy Density

Under standard physical assumptions $F$ will have $m + 1$ local minima, and by subtracting a supporting hyperplane we can take $F$ to be 0 at each of these points. A standard example is

$$F(u_1, u_2) = u_1^2 u_2^2 + u_1^2 (1 - u_1 - u_2)^2 + u_2^2 (1 - u_1 - u_2)^2.$$  

![Diagram of the bulk free energy function.](image)

**Figure:** Example bulk free energy function.

A transition front is a stationary solution $\bar{u}(x_1)$ that connects one of these minima to another.
The Bulk Free Energy Density

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A transition front is a stationary solution $\bar{u}(x_1)$ that connects one of these minima to another.
Transition Fronts

Transition fronts $\bar{u}(x_1)$ solve the equation

$$0 = \left( M(\bar{u})(-\Gamma \bar{u}'' + DF(\bar{u}))' \right)' ,$$

with ($u_+ \neq u_-$)

$$\lim_{x_1 \to -\infty} \bar{u}(x_1) = u_- \text{ (with } DF(u_-) = 0)$$

$$\lim_{x_1 \to +\infty} \bar{u}(x_1) = u_+ \text{ (with } DF(u_+) = 0)$$

$$\lim_{x_1 \to \pm\infty} \bar{u}'(x_1) = 0 .$$

Integrating twice, we find

$$-\Gamma \bar{u}'' + DF(\bar{u}) = 0 .$$
Example Case

For \( M(u) \equiv I, \Gamma = I, \) and
\[
F(u) = u_1^2 u_2^2 + u_1^2 (1 - u_1 - u_2)^2 + u_2^2 (1 - u_1 - u_2)^2.
\]

Figure: Binary Transition Front.
Existence of Transition Fronts

The existence of transition fronts for Cahn-Hilliard systems has been established under quite general conditions by:

- N. D. Alikakos, S. I. Betelu, and X. Chen (2006): for $m = 2$, using complex analysis
- N. D. Alikakos and G. Fusco (2008): $m \geq 2$, for $\Gamma = I$
- V. Stefanopoulos (2008): $m \geq 2$, for $\Gamma$ positive definite and symmetric

In each of these references, the transition fronts arise as minimizers of the associated energy functional

$$E(u) = \int_{-\infty}^{+\infty} F(u) + \frac{1}{2} \langle u_x, \Gamma u_x \rangle \, dx.$$
Linearization

Linearizing our Cahn-Hilliard equation about $\bar{u}$ ($u = \bar{u} + v$), we obtain

$$\nu_t = Lv,$$

with

$$Lv = \nabla \cdot \left\{ \tilde{M}(x_1) D_x \left( -\Gamma \Delta v + \tilde{B}(x_1) v \right) \right\},$$

where

$$\tilde{M}(x_1) = M(\bar{u}(x_1))$$
$$\tilde{B}(x_1) = D^2_u F(\bar{u}(x_1)).$$

The associated eigenvalue problem is

$$L\phi = \lambda\phi.$$
Fourier Transform

We take a Fourier transform in the transverse variable \( \tilde{x} = (x_2, x_3, \ldots, x_n) \) (\( \tilde{x} \to \xi \)) to obtain

\[
L_{\xi} \hat{\phi} = -A_{\xi} H_{\xi} \hat{\phi},
\]

where

\[
A_{\xi} := -\partial_{x_1} \bar{M}(x_1) \partial_{x_1} + |\xi|^2 \bar{M}(x_1)
\]
\[
H_{\xi} := -\Gamma \partial^2_{x_1x_1} + \bar{B}(x_1) + |\xi|^2 \Gamma.
\]

We make assumptions on our original equation that guarantee \( \bar{B}, \bar{M} \in C^2(\mathbb{R}) \), and that as \( x_1 \to \pm \infty \) they approach asymptotic endstates \( B_{\pm} \) and \( M_{\pm} \) at exponential rate. We assume \( B_{\pm} \) are positive definite, and \( \bar{M}(x_1) \) is uniformly positive definite.
Assumptions on \( \bar{u} \)

In addition to our technical assumptions, we make two assumptions on the nature of our wave \( \bar{u} \).

**Assumption 1.** We assume that \( \bar{u} \) minimizes the energy functional

\[
E(u) = \int_{-\infty}^{+\infty} F(u) + \frac{1}{2} \langle \Gamma u_{x_1}, u_{x_1} \rangle dx_1.
\]

**Assumption 2.** We assume that when

\[
-\Gamma \bar{u}'' + D_u F(\bar{u}) = 0
\]

is viewed as a first order system \( \bar{u} \) arises as a transverse connection from the \( m \)-dimensional unstable subspace at \( u_- \) to the \( m \)-dimensional stable subspace at \( u_+ \) (or vice versa, by isotropy).
Notes on our Assumptions

For **Assumption 1**, we recall that transition fronts naturally arise as minimizers of this energy.

Regarding **Assumption 2**, transversality can be linked to an appropriate Evans function condition derived by H. and Bongsuk Kwon [*Spectral analysis for transition front solutions in Cahn-Hilliard systems*, Discrete and Continuous Dynamical Systems A 32 (2012) 126-166].
Theorem, Part 1

Our theorem describes the spectrum of $L_\xi = -A_\xi H_\xi$. We establish:

0. $L_0 \bar{u}' = 0$. (In fact, $H_0 \bar{u}' = 0$.)

1. The spectrum $\sigma(L_\xi)$ lies entirely on $\mathbb{R}$.

2. The essential spectrum of $L_\xi$ lies in the union of the two intervals

   $$(-\infty, -m_\pm b_\pm |\xi|^2 - m_\pm \gamma |\xi|^4],$$

   where $m_\pm$, $b_\pm$, and $\gamma$ respectively denote the smallest eigenvalues of $M_\pm$, $B_\pm$, and $\Gamma$.

3. There exists a constant $\theta_0 > 0$ so that the point spectrum of $L_\xi$ is confined to the interval

   $$(-\infty, -\theta_0 |\xi|^4].$$
4. For $|\xi|$ sufficiently small, the leading eigenvalue of $L_\xi$ satisfies

$$\lambda_*(\xi) = -c_3|\xi|^3(1 + o(|\xi|)),$$

where

$$c_3 = 4 \left( \frac{\int_{-\infty}^{+\infty} F(\bar{u}(x_1)) dx_1}{\langle M^{-1}[u], [u] \rangle} \right) > 0.$$

Here, $[u] = u_+ - u_-.$

5. For $|\xi|$ sufficiently small there exists a constant $\theta_1 > 0$ so that the set $\sigma_{pt}(L_\xi) \setminus \{\lambda_*(\xi)\}$ is confined to the interval

$$(-\infty, -\theta_1|\xi|^2].$$
Spectrum for $L_0$

![Diagram showing the spectrum for $L_0$ with essential spectrum and $\lambda_*(0)$ marked.](image-url)
Spectrum for $L_\xi$, $\xi \neq 0$
Spectrum for $L_\xi$, $\xi \neq 0$

Possible additional eigenvalues $\leq -\theta_1|\xi|^2$

Essential spectrum

$\lambda_{\text{ess}} < -m \pm b \pm |\xi|^2$

$\lambda_* (\xi) \sim -c_3|\xi|^3$
Remark on Nonlinear Stability

For the single Cahn-Hilliard equation

\[ u_t = \Delta(-\Delta u + \frac{1}{2}u^3 - \frac{1}{2}u) \]

on \( \mathbb{R}^n, n \geq 3 \), Korvola, Kupiainen, and Taskinen have established that this spectral behavior implies nonlinear stability. [Anomalous scaling for three-dimensional Cahn-Hilliard fronts, CPAM LVIII (2005) 1-39].

Nonlinear stability was established for \( n \geq 2 \) and a general class of \( M(u) \) and \( F(u) \) by H. [Asymptotic behavior near planar transition fronts for equations of Cahn-Hilliard type, Physica D 229 (2007) 123-165].

For systems, nonlinear stability (currently) requires an additional technical assumption on the Evans function.
Elements of the Proof

We recall that our operator is

$$L_\xi = -A_\xi H_\xi,$$

where

$$A_\xi := -\partial_{x_1} \tilde{M}(x_1) \partial_{x_1} + |\xi|^2 \tilde{M}(x_1)$$

$$H_\xi := -\Gamma \partial_{x_1 x_1}^2 + \bar{B}(x_1) + |\xi|^2 \Gamma.$$

For $\xi \neq 0$ we have that $A_\xi^{-1}$ is a bounded, self-adjoint, positive definite operator. Setting $\varphi := A_\xi^{-1/2} \hat{\varphi}$, our eigenvalue equation becomes

$$L_\xi \varphi := -A_\xi^{1/2} H_\xi A_\xi^{1/2} \varphi = \lambda \varphi.$$

The operator $L_\xi$ is self-adjoint, allowing the application of minimax techniques, which give Parts (3) and (5) of our theorem, and also partly justify the proof of Part (4).
Focus on $\lambda_*(\xi)$

We’ve already observed that

$$H_0 \bar{u}' = 0,$$

and it follows (recall $H_\xi = H_0 + |\xi|^2 \Gamma$) that

$$H_\xi \bar{u}' = |\xi|^2 \Gamma \bar{u}' .$$

Next, we express our original eigenvalue problem as

$$-H_\xi \hat{\phi} = \lambda A_\xi^{-1} \hat{\phi} ,$$

and if we denote the eigenfunction for $\lambda_*$ by $\hat{\phi}_*$

$$-H_\xi \hat{\phi}_* = \lambda_* A_\xi^{-1} \hat{\phi}_* .$$

We take an inner product of this equation with $\bar{u}'$, and recall that $H_\xi$ is self-adjoint.
Focus on $\lambda_*(\xi)$

This gives

$$-\langle \hat{\phi}_*, H_{\xi} \bar{u}' \rangle = \lambda_* \langle A_{\xi}^{-1} \hat{\phi}_*, \bar{u}' \rangle.$$ 

We use $H_{\xi} \bar{u}' = |\xi|^2 \Gamma \bar{u}'$ and (now take $\bar{M}$ constant)

$$A_{\xi}^{-1} \hat{\phi}_* = \frac{1}{2|\xi|} \int_{-\infty}^{+\infty} e^{-|\xi||x_1 - y_1|} \bar{M}^{-1} \hat{\phi}_*(y_1; |\xi|)dy_1.$$ 

We see that

$$\lambda_*(\xi) = -2|\xi|^3 \frac{\langle \hat{\phi}_*, \Gamma \bar{u}' \rangle}{\langle \int_{-\infty}^{+\infty} e^{-|\xi||x_1 - y_1|} \bar{M}^{-1} \hat{\phi}_*(y_1; |\xi|)dy_1, \bar{u}' \rangle}.$$ 

We obtain our claimed behavior on $\lambda_*$ if we can justify a perturbation analysis, including

$$\lim_{|\xi| \to 0} \hat{\phi}_*(x_1; |\xi|) = \bar{u}'.$$
Justifying the Perturbation Argument

Our goal is to perturb from $\lambda_*(0) = 0$, but we keep in mind that for $\xi = 0$ the essential spectrum is $(-\infty, 0]$, so $\lambda_*$ is not isolated.

Recalling that $H_0 \bar{u}' = 0$, we have

$$-A_\xi H_0 \bar{u}' = 0,$$

where $\lambda = 0$ is an isolated eigenvalue of $-A_\xi H_0$.

In addition, we can verify that $\lambda = 0$ is a simple eigenvalue of $-A_\xi H_0$. The condition for this is a transversality condition inherited from the associated analysis for $x \in \mathbb{R}$. 

Transversality Condition

Focusing on $\lambda_*(0)$, the case $\xi = 0$ corresponds with the operator $L_0$, and the eigenvalue problem

$$\left( M(\bar{u})(-\Gamma \phi_{xx} + D_u F(\bar{u})\phi)x \right)_x = \lambda \phi.$$  

This equation has $2m$ solutions that decay as $x \to -\infty$, $\{\phi_j^-\}_{j=1}^{2m}$ and $2m$ solutions that decay as $x \to +\infty$, $\{\phi_j^+\}_{j=1}^{2m}$. We will set

$$\Phi^\pm := \left( \begin{array}{c} \phi_j^\pm \\ \phi_j^\pm' \\ \phi_j^\pm'' \\ \phi_j^\pm''' \end{array} \right), \quad \Phi^\pm := (\Phi_1^\pm, \ldots, \Phi_{2m}^\pm).$$

The Evans function is

$$D(\lambda) = \det(\Phi^+(0; \lambda), \Phi^-(0; \lambda)).$$
Characteristics of the Evans Function

Recall that any eigenfunction $\phi$ must be in $H^2$, and so it must decay at both $\pm\infty$. In this way, there must exist constants $\{\alpha_j\}_{j=1}^{2m}$ and $\{\beta_j\}_{j=1}^{2m}$ so that

$$
\sum_{j=1}^{2m} \alpha_j \phi_j^-(x; \lambda) = \phi(x; \lambda) = \sum_{j=1}^{2m} \beta_j \phi_j^+(x; \lambda).
$$

By linear dependence $D(\lambda) = 0$. We know $\lambda = 0$ is an eigenvalue, and it follows that $D(0) = 0$.

We would like to characterize this eigenvalue further by computing $D'(0)$, $D''(0)$, etc. However, $D$ is not differentiable at $\lambda = 0$. 
Analyticity of the Evans Function

The solutions \( \{ \phi_j^- \}_{j=1}^{2m} \) and \( \{ \phi_j^+ \}_{j=1}^{2m} \) have the form

\[
\phi_j^-(x; \lambda) = e^{\mu_{2m+j}^-} (r_{2m+j}^- + O(e^{-\eta|x|}))
\]

\[
\phi_j^+(x; \lambda) = e^{\mu_{j}^+} (r_{j}^+ + O(e^{-\eta|x|}))
\]

where for \( j = 1, \ldots, m \)

\[
\mu_{j}^\pm (\lambda) = -\sqrt{\nu_{m+1-j}^\pm} + O(|\lambda|)
\]

\[
\mu_{m+j}^\pm (\lambda) = -\sqrt{\frac{\lambda}{\beta_j^\pm}} + O(|\lambda|^{3/2})
\]

\[
\mu_{2m+j}^\pm (\lambda) = \sqrt{\frac{\lambda}{\beta_{m+1-j}^\pm}} + O(|\lambda|^{3/2})
\]

\[
\mu_{3m+j}^\pm (\lambda) = \sqrt{\nu_j^\pm} + O(|\lambda|).
\]
The Stability Condition

We can view the Evans function as an analytic function of \( \rho = \sqrt{\lambda} \). We denote this function \( D_a(\rho) \). It is straightforward to verify that

\[
D_a(0) = D'_a(0) = \cdots = D^{(m)}_a(0) = 0.
\]

Our transversality condition (guaranteeing that \( \lambda = 0 \) is simple) is

\[
\frac{d^{m+1}D_a}{d\rho^{m+1}}(0) \neq 0.
\]

For \( m = 1 \) (the case of a single equation) it's easy to verify that this holds under standard physical assumptions. For \( m \geq 2 \), we have developed a framework for verifying this condition on a case-by-case basis.

Note. Nonlinear stability requires an additional technical assumption on the Evans function for \( |\xi| \neq 0 \).
For $m = 2$ (two equations), the Evans function $D_a(\rho)$ (for $|\xi| = 0$) satisfies

$$D_a'''(0) = c_1 + c_2 + c_3 + c_4,$$

where we are simply designating that it is the sum of four terms that can be computed individually. The condition $D_a'''(0) \neq 0$ asserts that for $|\lambda|$ small

$$D(\lambda) = (c_1 + c_2 + c_3 + c_4)\lambda^{3/2} + O(|\lambda|^2).$$
Technical Assumption

For \(|\xi| \neq 0\) we find that the analogous expression is

\[
D(\lambda, \xi) = \lambda \left( c_1 \sqrt{\lambda + \beta_1^- |\xi|^2} + c_2 \sqrt{\lambda + \beta_2^- |\xi|^2} \\
+ c_3 \sqrt{\lambda + \beta_1^+ |\xi|^2} + c_4 \sqrt{\lambda + \beta_2^+ |\xi|^2} \right) \\
+ O\left((|\lambda|^2 + |\xi|^4)\right).
\]

In this case, we need a stronger assumption such as: each of the \(c_j\) has the same sign.
Nonlinear Stability Theorem

Suppose \( \tilde{u}(x_1) \) is a planar transition front for which all spectral conditions are satisfied. Then for Hölder continuous initial conditions \( u_0(x) \in C^\gamma(\mathbb{R}^n) \), \( 0 < \gamma < 1 \), with

\[
\| u(0, x) - \tilde{u}(x_1) \|_{L^1_x} + \| u(0, x) - \tilde{u}(x_1) \|_{L^\infty_x} \leq \epsilon,
\]

for \( \epsilon > 0 \) sufficiently small, there exists a unique solution of (CH)

\[
u \in C^{4+\gamma, 1+\frac{\gamma}{4}}(\mathbb{R}^n \times (0, \infty)) \cap C^{\gamma, \frac{\gamma}{4}}(\mathbb{R}^n \times [0, \infty))
\]

and a shift \( \delta \in C^{3+\gamma, 1+\gamma}(\mathbb{R}^{n-1} \times [0, \infty)) \) so that

\[
\| u(x, t) - \tilde{u}(x_1 - \delta(\tilde{x}, t)) \|_{L^p_x} \leq C\epsilon \left[ (1 + t)^{-\frac{n}{2}(1 - \frac{1}{p})} + (1 + t)^{-\frac{n-1}{3}(1 - \frac{1}{p}) - \frac{2}{3} + \frac{1}{2p}} h_{n,p}(t) \right]
\]

\[
\| \delta(\tilde{x}, t) \|_{L^p_{\tilde{x}}} \leq C\epsilon (1 + t)^{-\frac{n-1}{3}(1 - \frac{1}{p})} h_{n,p}(t).
\]
The Aggravation Function $h_{n,p}(t)$

The function $h_{n,p}$ is defined for all $n = 2, 3, \ldots$ and $1 \leq p \leq \infty$. We define $h_{2,p}(t) = 1$ for all $p \in [1, \infty]$, and for $n = 3, 4, \ldots$

$$h_{n,p}(t) = \begin{cases} \ln(e + t) & p = 1 \\ 1 & p > 1. \end{cases}$$