Suppose that \( x^{(1)}(t), \ldots, x^{(n)}(t) \) form a fundamental set of solutions to the homogeneous linear systems of equations with constant coefficients
\[
x' = Ax,
\]
on some interval \( I \). Then the matrix
\[
X(t) = \begin{pmatrix} x^{(1)}(t) & \cdots & x^{(n)}(t) \\
x_{1,1}(t) & \cdots & x_{1,n}(t) \\
\vdots & \ddots & \vdots \\
x_{n,1}(t) & \cdots & x_{n,n}(t) \end{pmatrix},
\]
whose columns are the vector functions \( x^{(1)}(t), \ldots, x^{(n)}(t) \) is called a fundamental matrix for the system of equations \( (I) \). The fundamental matrix is always nonsingular since \( W[x^{(1)}, \ldots, x^{(n)}](t) \neq 0 \) in the interval \( I \).

**Example:** Find a fundamental matrix for the system
\[
x' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} x.
\]
We can now write the solution of an initial value problem in compact form. First, we note that the general solution to (1) is written
\[ x(t) = c_1 x^{(1)}(t) + \ldots + x^{(n)}(t) = X(t)c, \]
with \( c = (c_1, \ldots, c_n)^T \). Suppose that an initial condition is given
\[ x(t_0) = x_0, \]
where \( t_0 \) is in the solution interval \( I \). Then by combining the two equations above we see that
\[ x(t_0) = X(t_0)c = x_0, \]
and hence the solution to the IVP \( x' = Ax \) with \( x(t_0) = x_0 \) is now
\[ \Phi(t) = X(t)X^{-1}(t_0). \]

**Example:** Solve the following initial value problem using the fundamental matrix.
\[ x' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \]
The Matrix Exponential

Remember that we know the solution to the scalar initial value problem

\[ x' = ax, \quad x(0) = x_0. \]

Now consider the corresponding IVP for an \( n \times n \) system of first order equations:

\[ x' = Ax, \quad x(t_0) = x_0. \]

From what we did above, the solution is

\[ x(t) = \Phi(t)x_0, \]

and hence we call \( \Phi(t) \) the **matrix exponential**, and denote it as \( \Phi(t) = e^{At} \).

**Example**: Consider the IVP

\[ x' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}. \]

(a) Find a fundamental matrix for the system.
(b) Find the matrix exponential $\Phi(t) = e^{At}$ and solve the IVP.